IMPULSE RESPONSE
Finding the Impulse Response of a linear time-invariant discrete-time (LTID) System

THE PROBLEM
Find the impulse response $h[k]$ of a system described by the equation:

$$(E^2 - 6E + 9)y[k] = (E + 18)f[k]$$

where $f[k]$ is the input and $y[k]$ is the output.

THE SOLUTION
From the equation given we can determine:

- Characteristic polynomial: $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$
- Characteristic roots: $\lambda = 3, 3$
- Characteristic modes: $3^k, k3^k$
- Zero-Input response: $y_0[k] = c_13^k + c_2k3^k$

Since $h[k]$ is the response to $\delta[k]$, the impulse function, substitute $h[k]$ for $y[k]$ and $\delta[k]$ for $f[k]$, and solve for $h[k]$.

$$(E^2 - 6E + 9)h[k] = (E + 18)\delta[k]$$

$$h[k] = \frac{E + 18}{E^2 - 6E + 9}\delta[k]$$

This is in the form $h[k] = \frac{b_0}{a_0}\delta[k]$ and for some reason we ignore the $E$ values to get:

$b_0 = 18$
$a_0 = 9$

$u[k]$ is a unit step function which equals 1 for $k \geq 0$, and equals 0 for $k < 0$. The initial condition is multiplied by $u[k]$ because $h[k]$ is causal. So for the form $h[k] = \frac{b_0}{a_0}\delta[k] + y_0[k]u[k]$ we have:

$$h[k] = 2\delta[k] + (c_13^k + c_2k3^k)u[k]$$

To find $c_1$ and $c_2$ we need 2 values of $h[k]$. Returning to equation (A):

$$(E^2 - 6E + 9)h[k] = (E + 18)\delta[k]$$

Since $E^3$ represents $k + 2$ and $E$ represents $k + 1$, we rewrite:

$$h[k + 2] - 6h[k + 1] + 9h[k] = \delta[k + 1] + 18\delta[k]$$

For $k = -2$ we have:

$$h[0] - 6h[-1] + 9h[-2] = \delta[-1] + 18\delta[-2]$$
We know that $h[-1] = h[-2] = 0$ and that $\delta[k] = 0$ except when $k = 0$, $\delta[0] = 1$, so:

<table>
<thead>
<tr>
<th>$h[0]$</th>
<th>$h[0] - 6(0) + 9(0) = 0 + 18(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h[0]$</td>
<td>$h[0] = 0$</td>
</tr>
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</table>

For $k = -1$, using equation (C):
we have:

<table>
<thead>
<tr>
<th>$h[k]$</th>
<th>$h[k + 2] - 6h[k + 1] + 9h[k] = \delta[k + 1] + 18\delta[k]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h[1]$</td>
<td>$h[1] - 6h[0] + 9h[-1] = \delta[0] + 18\delta[-1]$</td>
</tr>
<tr>
<td>$h[1]$</td>
<td>$h[1] - 6(0) + 9(0) = 1 + 18(0)$</td>
</tr>
<tr>
<td>$h[1]$</td>
<td>$h[1] = 1$</td>
</tr>
</tbody>
</table>

Using equation (B):

For $k = 0$ we have:

- $0 = 2 + [c_1 + c_2(0)]$ 1
- $0 = 2 + c_1$
- $c_1 = -2$

For $k = 1$ we have:

- $1 = 2(0) + [3c_1 + 3c_2]$ 1
- $1 = 3(-2) + 3c_2$
- $7 = 3c_2$
- $c_2 = \frac{7}{3}$

**THE ANSWER**

Substituting into equation (B):

$\begin{align*}
  h[k] &= 2\delta[k] + \left([(-2)3^k + \frac{7}{3}k3^k]\right)u[k] \\
  h[k] &= 2\delta[k] - \left(2 - \frac{7}{3}k\right)3^k u[k]
\end{align*}$