ENGINEERING ACOUSTICS EE 363N

INDEX

DECIBELS [dB]

A log based unit of energy that makes it easier to describe exponential losses, etc. The decibel means 10 bels, a unit named after Bell Laboratories.

$$\boxed{L = 10\log\frac{\text{energy}}{\text{reference energy}}} \quad \text{[dB]}$$

One decibel is approximately the minimum discernable amplitude difference that can be detected by the human ear over the full range of amplitude.

l WAVELENGTH [m]

Wavelength is the distance that a wave advances during one cycle.

$$\boxed{\lambda = \frac{c}{f} = \frac{2\pi}{k}}$$

At **high temperatures**, the speed of sound increases so λ changes. T_k is temperature in Kelvin.

$$\lambda = \frac{343}{f}\sqrt{\frac{T_k}{293}}$$

k WAVE NUMBER [rad/m]

The wave number of propagation constant is a component of a wave function representing the wave density or wave spacing relative to distance. Sometimes represented by the letter β. See also WAVE VECTOR p9.

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{\omega}{c}}$$

s CONDENSATION [no units]

The ratio of the change in density to the ambient density, i.e. the degree to which the medium has condensed (or expanded) due to sound waves. For example, $s = 0$ means no condensation or expansion of the medium. $s = -½$ means the density is at one half the ambient value. $s = +1$ means the density is at twice the ambient value. Of course these examples are unrealistic for most sounds; the condensation will typically be close to zero.

$$\boxed{s = \frac{\rho - \rho_0}{\rho_0}}$$

ρ = instantaneous density [kg/m^3]
ρ_0 = equilibrium (ambient) density [kg/m^3]

UNITS

A (amp) = $\frac{C}{s}=\frac{W}{V}=\frac{J}{V\cdot s}=\frac{N\cdot m}{V\cdot s}=\frac{V\cdot F}{s}$

C (coulomb) = $A\cdot s=V\cdot F=\frac{J}{V}=\frac{N\cdot m}{V}=\frac{W\cdot s}{V}$

F (farad) = $\frac{C}{V}=\frac{C^2}{J}=\frac{C^2}{N\cdot m}=\frac{J}{V^2}=\frac{A\cdot s}{V}$

H (henry) = $\frac{V\cdot s}{A}$ (note that $\mathrm{H}\cdot\mathrm{F}=s^2$)

J (joule) = $N\cdot m=V\cdot C=W\cdot s=A\cdot V\cdot s=F\cdot V^2=\frac{C^2}{F}$

N (newton) = $\frac{J}{m}=\frac{C\cdot V}{m}=\frac{W\cdot s}{m}=\frac{kg\cdot m}{s^2}$

Pa (pascal) = $\frac{N}{m^2}=\frac{kg}{m\cdot s^2}=\frac{J}{m^3}=\frac{W\cdot s}{m^3}$

T (tesla) = $\frac{Wb}{m^2}=\frac{V\cdot s}{m^2}=\frac{H\cdot A}{m^2}$

V (volt) = $\frac{W}{A}=\frac{J}{C}=\frac{J}{A\cdot s}=\frac{W\cdot s}{C}=\frac{N\cdot m}{C}=\frac{C}{F}$

W (watt) = $\frac{J}{s}=\frac{N\cdot m}{s}=\frac{C\cdot V}{s}=V\cdot A=\frac{F\cdot V^2}{s}=\frac{1}{746}HP$

Wb (weber) = $H\cdot A=V\cdot s=\frac{J}{A}$

Acoustic impedance: [rayls or (Pa · s)/m]

Temperature: [°C or K] 0°C = 273.15K

c SPEED OF SOUND [m/s]

Sound travels faster in stiffer (i.e. higher $\mathscr{B}$, less compressible) materials. Sound travels faster at higher temperatures.

Frequency/wavelength relation: $c=\lambda f=\frac{\lambda\omega}{2\pi}$

In a perfect gas: $c=\sqrt{\frac{\gamma\mathscr{P}_0}{\rho_0}}=\sqrt{\gamma r T_K}$

In liquids: $c=\sqrt{\frac{\gamma\mathscr{B}_T}{\rho_0}}$ where $\mathscr{B}=\gamma\mathscr{B}_T$

γ = ratio of specific heats (1.4 for a diatomic gas) [no units]

$\mathscr{P}_0$ = ambient (atmospheric) pressure ($p \ll \mathscr{P}_0$). At sea level, $\mathscr{P}_0 \approx 101\,\mathrm{kPa}$ [Pa]

ρ_0 = equilibrium (ambient) density [kg/m^3]

r = specific gas constant [J/(kg· K)]

T_K = temperature in Kelvin [K]

$\mathscr{B}=\rho_0\left(\frac{\partial\mathscr{P}}{\partial\rho}\right)_{\rho_0}$ adiabatic bulk modulus [Pa]

$\mathscr{B}_T$ = isothermal bulk modulus, easier to measure than the adiabatic bulk modulus [Pa]

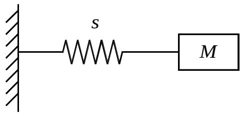
Two values are given for the speed of sound in solids, Bar and Bulk. The Bar value provides for the ability of sound to distort the dimensions of solids having a small-cross-sectional area. Sound moves more slowly in Bar material. The Bulk value is used below where applicable.

Speed of Sound in Selected Materials [m/s]

Air @ 20°C	343	Copper	5000	Steel	6100
Aluminum	6300	Glass (pyrex)	5600	Water, fresh 20°C	1481
Brass	4700	Ice	3200	Water, sea 13°C	1500
Concrete	3100	Steam @ 100°C	404.8	Wood, oak	4000

SIMPLE HARMONIC MOTION

Restoring force on a spring (Hooke's Law):



$$f_s = -sx$$

and Newton's Law:

$$F = ma$$

yield: $-sx = m\dfrac{d^2x}{dt^2}$ and $\dfrac{d^2x}{dt^2} + \dfrac{s}{m}x = 0$

Let $\omega_0{}^2 = \dfrac{s}{m}$, so that the system is described by the

equation $\boxed{\dfrac{d^2x}{dt^2} + \omega_0{}^2 x = 0}$.

$\omega_0 = \sqrt{\dfrac{s}{m}}$ is the *natural angular frequency* in rad/s.

$f_0 = \dfrac{\omega_0}{2\pi}$ is the *natural frequency* in Hz.

The general solution takes the form

$$x(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t$$

Initial conditions:

displacement: $x(0) = x_0$, so $A_1 = x_0$

velocity $\dot{x}(0) = u_0$, so $A_2 = \dfrac{u_0}{\omega_0}$

Solution: $\boxed{x(t) = x_0 \cos \omega_0 t + \dfrac{u_0}{\omega_0} \sin \omega_0 t}$

s = spring constant [no units]
x = the displacement [m]
m = mass [kg]
u = velocity of the mass [m/s]
t = time [s]

SIMPLE HARMONIC MOTION, POLAR FORM

The solution above can be written

$$\boxed{x(t) = A\cos(\omega_0 t + \phi)},$$

where we have the new constants:

amplitude: $A = \sqrt{x_0{}^2 + \left(\dfrac{u_0}{\omega_0}\right)^2}$

initial phase angle: $\phi = \tan^{-1}\left(\dfrac{-u_0}{\omega_0 x_0}\right)$

Note that zero phase angle occurs at maximum positive displacement.

By differentiation, it can be found that the speed of the mass is $u = -U\sin(\omega_0 t + \phi)$, where $U = \omega_0 A$ is the *speed amplitude*. The acceleration is $a = -\omega_0 U \cos(\omega_0 t + \phi)$.

Using the initial conditions, the equation can be written

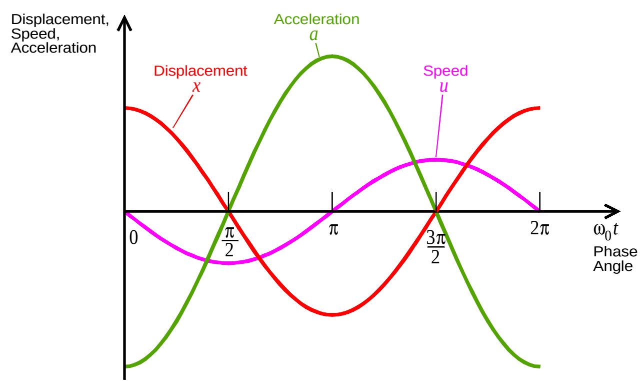
$$\boxed{x(t) = \sqrt{x_0{}^2 + \left(\dfrac{u_0}{\omega_0}\right)^2} \cos\left(\omega_0 t - \tan^{-1}\dfrac{u_0}{\omega_0 x_0}\right)}$$

x_0 = the initial position [m]
u_0 = the initial speed [m/s]

$\omega_0 = \sqrt{\dfrac{s}{m}}$ is the *natural angular frequency* in rad/s.

It is seen that displacement lags 90° behind the speed and that the acceleration is 180° out of phase with the displacement.

SIMPLE HARMONIC MOTION, displacement – acceleration - speed



Initial phase angle f=0°

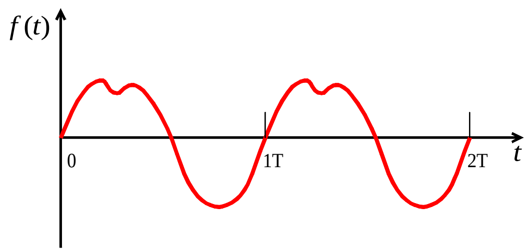
The speed of a simple oscillator leads the displacement by 90°. Acceleration and displacement are 180° out of phase with each other.

FOURIER SERIES

The Fourier Series is a method of describing a complex periodic function in terms of the frequencies and amplitudes of its fundamental and harmonic frequencies.

Let $f(t) = f(t+T) =$ any periodic signal

where $T =$ the period.



Then $\boxed{f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos n\omega t + B_n \sin n\omega t\right)}$

where $\boxed{\omega = \frac{2\pi}{T}}$ = the fundamental frequency

A_0 = the DC component and will be zero provided the function is symmetric about the t-axis. This is almost always the case in acoustics.

$A_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega t \, dt$

A_n is zero when $f(t)$ is an **odd function**, i.e. $f(t)=-f(-t)$, the right-hand plane is a mirror image of the left-hand plane provided one of them is first flipped about the horizontal axis, e.g. sine function.

$B_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega t \, dt$

B_n is zero when $f(t)$ is an **even function**, i.e. $f(t)=f(-t)$, the right-hand plane is a mirror image of the left-hand plane, e.g. cosine function.

where $t_0 =$ an arbitrary time

p ACOUSTIC PRESSURE [Pa]

Sound waves produce proportional changes in pressure, density, and temperature. Sound is usually measured as a change in pressure. See Plane Waves p9.

$$p = \mathscr{P} - \mathscr{P}_0$$

For a simple harmonic plane wave traveling in the x direction, p is a function of x and t:

$$p(x,t) = Pe^{\mathrm{j}(\omega t - kx)}$$

$\mathscr{P}$ = instantaneous pressure [Pa]

$\mathscr{P}_0$ = ambient (atmospheric) pressure ($p \ll \mathscr{P}_0$). At sea level, $\mathscr{P}_0 \approx 101\,\mathrm{kPa}$ [Pa]

P = peak acoustic pressure [Pa]

x = position along the x-axis [m]

t = time [s]

P_e EFFECTIVE ACOUSTIC PRESSURE [Pa]

The effective acoustic pressure is the rms value of the sound pressure, or the rms sum (see page 34) of the values of multiple acoustic sources.

$$\boxed{P_e = \frac{P}{\sqrt{2}}} \qquad P_e^2 = \langle P^2 \rangle = \int_T p^2 dt$$

$$P_e = \sqrt{\langle P_1^2 \rangle + \langle P_2^2 \rangle + \langle P_3^2 \rangle + \cdots}$$

P = peak acoustic pressure [Pa]

p = $\mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]

ρ_0 EQUILIBRIUM DENSITY [kg/m^3]

The ambient density.

$$\rho_0 = \frac{\mathscr{B}}{c^2} = \frac{\gamma \mathscr{P}_0}{c^2} \quad \text{for ideal gases}$$

$$\rho_0 = \frac{\gamma \mathscr{B}_T}{c^2} \quad \text{for liquids}$$

The equilibrium density is the inverse of the specific volume. From the ideal gas equation:

$$P\nu = RT \rightarrow P = \rho_0 RT$$

$\mathscr{B} = \rho_0 \left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_{\rho_0}$ adiabatic bulk modulus, approximately equal to the isothermal bulk modulus, 2.18×10^9 for water [Pa]

c = the **phase speed** (speed of sound) [m/s]

γ = ratio of specific heats (1.4 for a diatomic gas) [no units]

$\mathscr{P}_0$ = ambient (atmospheric) pressure ($p \ll \mathscr{P}_0$). At sea level, $\mathscr{P}_0 \approx 101\,\text{kPa}$ [Pa]

P = pressure [Pa]

ν = V/m specific volume [m^3/kg]

V = volume [m^3]

m = mass [kg]

R = gas constant (287 for air) [J/(kg· K)]

T = absolute temperature [K] (°C + 273.15)

ρ_0 Equilibrium Density of Selected Materials [kg/m^3]					
Air @ 20°C	1.21	Copper	8900	Steel	7700
Aluminum	2700	Glass (pyrex)	2300	Water, fresh 20°C	998
Brass	8500	Ice	920	Water, sea 13°C	1026
Concrete	2600	Steam @ 100°C	0.6	Wood, oak	720

$\mathscr{B}$ ADIABATIC BULK MODULUS [Pa]

$\mathscr{B}$ is a stiffness parameter. A larger $\mathscr{B}$ means the material is not as compressible and sound travels faster within the material.

$$\boxed{\mathscr{B} = \rho_0 \left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_{\rho_0} = \rho_0 c^2 = \gamma \mathscr{P}_0}$$

ρ = instantaneous density [kg/m^3]

ρ_0 = equilibrium (ambient) density [kg/m^3]

c = the **phase speed** (speed of sound, 343 m/s in air) [m/s]

$\mathscr{P}$ = instantaneous (total) pressure [Pa or N/m^2]

$\mathscr{P}_0$ = ambient (atmospheric) pressure ($p \ll \mathscr{P}_0$). At sea level, $\mathscr{P}_0 \approx 101\,\text{kPa}$ [Pa]

γ = ratio of specific heats (1.4 for a diatomic gas) [no units]

$\mathscr{B}$ Bulk Modulus of Selected Materials [Pa]					
Aluminum	75×10^9	Iron (cast)	86×10^9	Rubber (hard)	5×10^9
Brass	136×10^9	Lead	42×10^9	Rubber (soft)	1×10^9
Copper	160×10^9	Quartz	33×10^9	Water	$*2.18\times10^9$
Glass (pyrex)	39×10^9	Steel	170×10^9	Water (sea)	$*2.28\times10^9$

*$\mathscr{B}_T$, isothermal bulk modulus

EQUATION OVERVIEW

Equation of State (pressure)

$$\boxed{p = \mathscr{B} s}$$

Mass Conservation (density)

3-dimensional

$$\boxed{\frac{\partial s}{\partial t} + \vec{\nabla} \cdot \vec{u} = 0}$$

1-dimensional

$$\boxed{\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0}$$

Momentum Conservation (velocity)

3-dimensional

$$\boxed{\vec{\nabla} p + \rho_0 \frac{\partial \vec{u}}{\partial t} = 0}$$

1-dimensional

$$\boxed{\frac{\partial p}{\partial x} + \rho_0 \frac{\partial u}{\partial t} = 0}$$

From the above 3 equations and 3 unknowns (p, s, u) we can derive the **Wave Equation**

$$\boxed{\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}}$$

EQUATION OF STATE - GAS

An equation of state relates the physical properties describing the thermodynamic behavior of the fluid. In acoustics, the temperature property can be ignored.

In a **perfect adiabatic gas**, the thermal conductivity of the gas and temperature gradients due to sound waves are so small that no appreciable thermal energy transfer occurs between adjacent elements of the gas.

Perfect adiabatic gas: $\dfrac{\mathscr{P}}{\mathscr{P}_0} = \left(\dfrac{\rho}{\rho_0}\right)^\gamma$

Linearized: $\boxed{p = \gamma \mathscr{P}_0 s}$

$\mathscr{P}$ = instantaneous (total) pressure [Pa]

$\mathscr{P}_0$ = ambient (atmospheric) pressure ($p \ll \mathscr{P}_0$). At sea level, $\mathscr{P}_0 \approx 101\,\text{kPa}$ [Pa]

ρ = instantaneous density [kg/m^3]

ρ_0 = equilibrium (ambient) density [kg/m^3]

γ = ratio of specific heats (1.4 for a diatomic gas) [no units]

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]

$s = \dfrac{\rho - \rho_0}{\rho_0} \ll 1$ condensation [no units]

EQUATION OF STATE – LIQUID

An equation of state relates the physical properties describing the thermodynamic behavior of the fluid. In acoustics, the temperature property can be ignored.

Adiabatic liquid: $\boxed{p = \mathscr{B}s}$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]

$\mathscr{B} = \rho_0 \left(\frac{\partial \mathscr{P}}{\partial \rho}\right)_{\rho_0}$ adiabatic bulk modulus, approximately equal to the isothermal bulk modulus, 2.18×10[9] for water [Pa]

$s = \frac{\rho - \rho_0}{\rho_0} \ll 1$ condensation [no units]

r SPECIFIC GAS CONSTANT [J/(kg· K)]

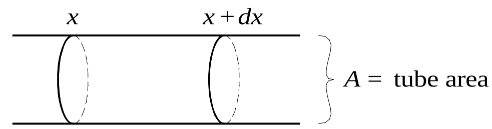
The specific gas constant r depends on the universal gas constant $\mathscr{R}$ and the molecular weight M of the particular gas. For air $r \approx 287 \text{ J/}(\text{kg}\cdot\text{K})$.

$$\boxed{r = \frac{\mathscr{R}}{M}}$$

$\mathscr{R}$ = universal gas constant
M = molecular weight

MASS CONSERVATION – one dimension

For the one-dimensional problem, consider sound waves traveling through a tube. Individual particles of the medium move back and forth in the x-direction.



$(\rho u A)_x$ is called the mass flux [kg/s]

$(\rho u A)_{x+dx}$ is what's coming out the other side (a different value due to compression) [kg/s]

The difference between the rate of mass entering the center volume ($A\,dx$) and the rate at which it leaves the center volume is the rate at which the mass is changing in the center volume.

$$(\rho u A)_x - (\rho u A)_{x+dx} = -\frac{\partial(\rho u A)}{\partial x}dx$$

$\rho\,dv$ is the mass in the center volume, so the rate at which the mass is changing can be written as

$$\frac{\partial}{\partial t}\rho\,dv = \frac{\partial}{\partial t}\rho A\,dx$$

Equating the two expressions gives

$$\frac{\partial}{\partial t}\rho A\,dx = -\frac{\partial(\rho u A)}{\partial x}dx$$, which can be simplified

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) = 0$$

u = particle velocity (due to oscillation, not flow) [m/s]
ρ = instantaneous density [kg/m^3]
$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
A = area of the tube [m^2]

MASS CONSERVATION – three dimensions

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

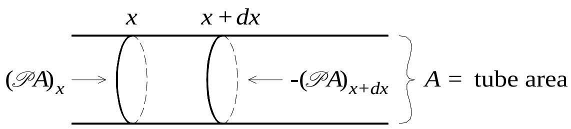
where $\vec{\nabla} = \rho\,\hat{x}\frac{\partial}{\partial x} + \rho\,\hat{y}\frac{\partial}{\partial y} + \rho\,\hat{z}\frac{\partial}{\partial z}$

and let $\rho = \rho_0(1+s)$

$$\boxed{\frac{\partial}{\partial t}s + \vec{\nabla}\cdot\vec{u} = 0}$$ (linearized)

MOMENTUM CONSERVATION – one dimension (5.4)

For the one-dimensional problem, consider sound waves traveling through a tube. Individual particles of the medium move back and forth in the *x*-direction.



$(\mathscr{P}A)_x$ is the force due to sound pressure at location x in the tube [N]

$(\mathscr{P}A)_{x+dx}$ is the force due to sound pressure at location $x + dx$ in the tube (taken to be in the positive or right-hand direction) [N]

The sum of the forces in the center volume is:

$$\sum F = (\mathscr{P}A)_x - (\mathscr{P}A)_{x+dx} = -A\frac{\partial \mathscr{P}}{\partial x}dx$$

Force in the tube can be written in this form, noting that this is not a partial derivative:

$$F = ma = (\rho A\,dx)\frac{du}{dt}$$

For some reason, this can be written as follows:

$$(\rho A\,dx)\frac{du}{dt} = (\rho A\,dx)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right)$$

with the term $u\frac{\partial u}{\partial x}$ often discarded in acoustics.

$\mathscr{P}$ = instantaneous (total) pressure [Pa or N/m^2]
A = area of the tube [m^2]
ρ = instantaneous density [kg/m^3]
$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
u = particle velocity (due to oscillation, not flow) [m/s]

MOMENTUM CONSERVATION – three dimensions

$$\frac{\partial}{\partial t}P + \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = 0$$

and $$\vec{\nabla}p + \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = 0$$

Note that $\vec{u}\cdot\vec{\nabla}\vec{u}$ is a quadratic term and that $\rho\frac{\partial \vec{u}}{\partial t}$ is quadratic after multiplication

$$\boxed{\vec{\nabla}p + \rho_0\frac{\partial \vec{u}}{\partial t} = 0}$$ (linearized)

ACOUSTIC ANALOGIES to electrical systems

	ACOUSTIC	ELECTRIC
Impedance:	$Z_A = \frac{p}{U}$	$Z = \frac{V}{I}$
Voltage:	Δp	$V = I\,R$
Current:	U	$I = \frac{V}{R}$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
U = volume velocity (not a vector) [m^3/s]
Z_A = acoustic impedance [Pa·s/m^3]

L INERTANCE [kg/m^4]

Describes the inertial properties of gas in a channel. Analogous to electrical inductance.

$$L = \frac{\rho_0 \Delta x}{A}$$

ρ_0 = ambient density [kg/m^3]
Δx = incremental distance [m]
A = cross-sectional area [m^2]

C COMPLIANCE [m^6/kg]

The *springiness* of the system; a higher value means *softer*. Analogous to electrical capacitance.

$$C = \frac{V}{\gamma\rho_0}$$

V = volume [m^3]
γ = ratio of specific heats (1.4 for a diatomic gas) [no units]
ρ_0 = ambient density [kg/m^3]

U VOLUME VELOCITY [m^3/s]

Although termed a *velocity*, volume velocity is not a vector. Volume velocity in a (uniform flow) duct is the product of the cross-sectional area and the velocity.

$$\boxed{U = \frac{\partial V}{\partial t} = \frac{d\xi}{dt}S = uS}$$

V = volume [m^3]
S = area [m^2]
u = velocity [m/s]
ξ = particle displacement, the displacement of a fluid element from its equilibrium position [m]

PLANE WAVES

PLANE WAVES (2.4, 5.7)

A disturbance a great distance from the source is approximated as a plane wave. Each acoustic variable has constant amplitude and phase on any plane perpendicular to the direction of propagation. The wave equation is the same as that for a disturbance on a string under tension.

There is no y or z dependence, so $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$.

One-dimensional wave equation: $\boxed{\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}}$

General Solution for the **acoustic pressure** of a plane wave:

$$\boxed{p(x,t) = \underbrace{Ae^{j(\omega t - kx)}}_{\text{propagating in the +x direction}} + \underbrace{Be^{j(\omega t + kx)}}_{\text{propagating in the -x direction}}}$$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
A = magnitude of the positive-traveling wave [Pa]
B = magnitude of the negative-traveling wave [Pa]
ω = frequency [rad/s]
t = time [s]
k = wave number or propagation constant [rad./m]
x = position along the x-axis [m]

PROGRESSIVE PLANE WAVE (2.8)

A progressive plane wave is a unidirectional plane wave—no reverse-propagating component.

$$\boxed{p(x,t) = Ae^{j(\omega t - kx)}}$$

ARBITRARY DIRECTION PLANE WAVE

The expression for an arbitrary direction plane wave contains wave numbers for the x, y, and z components.

$$p(x,t) = Ae^{j(\omega t - k_x x - k_y y - k_z z)}$$

where $k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2$

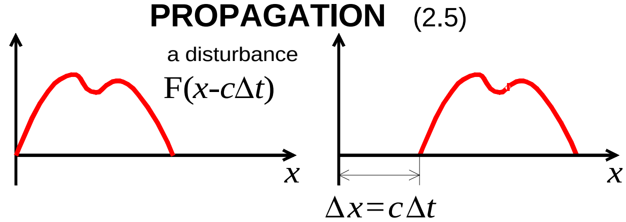
u VELOCITY, PLANE WAVE [m/s]

The acoustic pressure divided by the impedance, also from the momentum equation:

$$\frac{\partial p}{\partial x} + \rho_0 \frac{\partial u}{\partial t} = 0 \rightarrow \boxed{u = \frac{p}{z} = \frac{p}{\rho_0 c}}$$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
z = wave impedance [rayls or (Pa· s)/m]
ρ_0 = equilibrium (ambient) density [kg/m^3]
$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) [m/s]
k = wave number or propagation constant [rad./m]
r = radial distance from the center of the sphere [m]

PROPAGATION (2.5)



$$c = \frac{\Delta x}{\Delta t} \rightarrow \frac{dx}{dt},\ \Delta t \rightarrow 0$$

$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) at which F is translated in the $+x$ direction. [m/s]

$\vec{k}$ WAVE VECTOR [rad/m or m^{-1}]

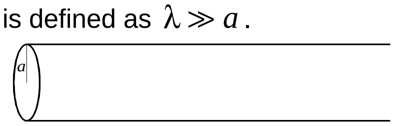
The phase constant k is converted to a vector. For plane waves, the vector $\vec{k}$ is in the direction of propagation.

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \quad \text{where}$$

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2$$

THIN ROD PROPAGATION

A thin rod is defined as $\lambda \gg a$.



a = rod radius

$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) [m/s]

Υ = Young's modulus, or modulus of elasticity, a characteristic property of the material [Pa]

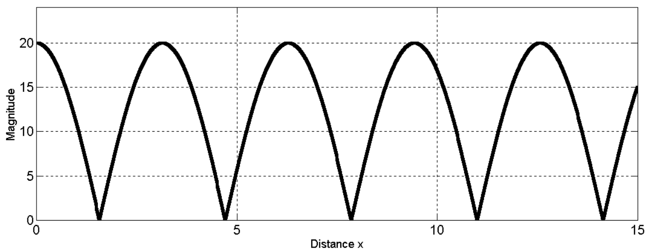
ρ_0 = equilibrium (ambient) density [kg/m^3]

STANDING WAVES

Two waves with identical frequency and phase characteristics traveling in opposite directions will cause constructive and destructive interference:

$$p(x,t) = \underbrace{p_1 e^{j(\omega t - kx)}}_{\text{moving right}} + \underbrace{p_2 e^{j(\omega t + kx)}}_{\text{moving left}}$$

for $p_1 = p_2 = 10$, $k=1$, $t=0$:



for $p_1 = 5$, $p_2 = 10$, $k=1$, $t=0$:

z SPECIFIC ACOUSTIC IMPEDANCE
[rayls or (Pa· s)/m] (5.10)

Specific acoustic impedance or **characteristic impedance** z is a property of the medium and of the type of wave being propagated. It is useful in calculations involving transmission from one medium to another. In the case of a plane wave, z is real and is independent of frequency. For spherical waves the opposite is true. In general, z is complex.

$$\boxed{z = \frac{p}{u} = \rho_0 c}$$ (applies to progressive plane waves)

Acoustic impedance is analogous to electrical impedance:

$$\frac{\text{pressure}}{\text{velocity}} = \text{impedance} = \frac{\text{volts}}{\text{amps}}$$

$$z = r + jx$$

In a sense this is resistive, i.e. a loss since the wave departs from the source.

In a sense this is reactive, in that this value represents an impediment to propagation.

$\rho_0 c$ Characteristic Impedance, Selected Materials (bulk) [rayls]

Air @ 20°C	415	Copper	44.5×10^6	Steel	47×10^6
Aluminum	17×10^6	Glass (pyrex)	12.9×10^6	Water, fresh 20°C	1.48×10^6
Brass	40×10^6	Ice	2.95×10^6	Water, sea 13°C	1.54×10^6
Concrete	8×10^6	Steam @ 100°C	242	Wood, oak	2.9×10^6

ξ PARTICLE DISPLACEMENT [m]

The displacement of a fluid element from its equilibrium position.

$$\boxed{\vec{u} = \frac{\partial \vec{\xi}}{\partial t}} \qquad \xi = \frac{p}{\omega \rho_0 c}$$

$\vec{u}$ = particle velocity [m/s]

P ACOUSTIC POWER [W]

Acoustic power is usually small compared to the power required to produce it.

$$\boxed{\Pi = \int_S \vec{I} \cdot d\vec{s}}$$

S = surface surrounding the sound source, or at least the surface area through which all of the sound passes [m^2]
I = acoustic intensity [W/m^2]

I ACOUSTIC INTENSITY [W/m^2]

The time-averaged rate of energy transmission through a unit area normal to the direction of propagation; power per unit area. Note that $I = \langle pu \rangle_T$ is a nonlinear equation (It's the product of two functions of space and time.) so you can't simply use $j\omega t$ or take the real parts and multiply, see *Time-Average* p33.

$$\boxed{I = \langle I(t) \rangle_T = \langle pu \rangle_T = \frac{1}{T}\int_0^T pu \, dt}$$

For a single frequency:

$$\boxed{I = \frac{1}{2} \mathscr{Re}\{p\,u^*\}}$$

For a plane harmonic wave traveling in the +z direction:

$$I = \frac{1}{A}\frac{\partial E}{\partial t} = \frac{1}{A}\frac{\partial E}{\partial x}\frac{\partial x}{\partial t} = c\frac{\partial E}{\partial V}, \qquad I = \frac{|p|^2}{2\rho_0 c} = \frac{P_e^2}{\rho_0 c}$$

T = period [s]
$I(t)$ = instantaneous intensity [W/m^2]
$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
$|p|$ = peak acoustic pressure [Pa]
u = particle velocity (due to oscillation, not flow) [m/s]
P_e = effective or rms acoustic pressure [Pa]
ρ_0 = equilibrium (ambient) density [kg/m^3]
$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) [m/s]

FOURIER'S LAW FOR HEAT CONDUCTION, HEAT FLUX

Sound waves produce proportional changes in pressure, density, and temperature. Since the periodic change in temperature is spread over the length of a wavelength, the change in temperature per unit distance is very small.

$$q = -K \frac{\partial T}{\partial x}$$

q = heat flux [°C/m]
T = temperature [°C]
K = a constant
x = distance [m]

SPHERICAL WAVES

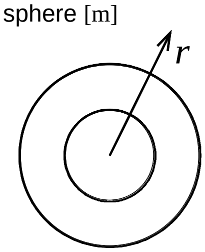
SPHERICAL WAVES (5.11)

General solution for a symmetric spherical wave:

$$\boxed{p(r,t) = \underbrace{\frac{A}{r} e^{\mathrm{j}(\omega t - kr)}}_{\text{diverging from the source}} + \underbrace{\frac{B}{r} e^{\mathrm{j}(\omega t + kr)}}_{\text{converging on the source}}}$$

Progressive spherical wave: $\boxed{p(r,t) = \frac{A}{r} e^{\mathrm{j}(\omega t - kr)}}$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
r = radial distance from the center of the sphere [m]
A = magnitude of the positive-traveling wave [Pa]
B = magnitude of the negative-traveling wave [Pa]
ω = frequency [rad/s]
t = time [s]
k = wave number or propagation constant [rad./m]



SPHERICAL WAVE BEHAVIOR

Spherical wave behavior changes markedly for very small or very large radii. Since this is also a function of the wavelength, we base this on the kr product where $kr \propto r/\lambda$.

For $kr \gg 1$, i.e. $r \gg \lambda$ (far from the source):
In this case, the spherical wave is much like a plane wave with the impedance $z \simeq \rho_0 c$ and with p and u in phase.

For $kr \ll 1$, i.e. $r \ll \lambda$ (close to the source):
In this case, the impedance is almost purely reactive $z \simeq \mathrm{j}\omega\rho_0 r$ and p and u are 90° out of phase. The source is not radiating power; particles are just sloshing back and forth near the source.

IMPEDANCE [rayls or (Pa· s)/m]

Spherical wave impedance is frequency dependent:

$$\boxed{z = \frac{p}{u} = \frac{\rho_0 c}{1 - \mathrm{j}/(kr)}}$$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
u = particle velocity (due to oscillation, not flow) [m/s]
ρ_0 = equilibrium (ambient) density [kg/m^3]
$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) [m/s]
k = wave number or propagation constant [rad./m]
r = radial distance from the center of the sphere [m]

u VELOCITY, SPHERICAL WAVE [m/s]

$$\boxed{u = \frac{p}{z} = \frac{p}{\rho_0 c}\left(1 - \frac{\mathrm{j}}{kr}\right)}$$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
z = wave impedance [rayls or (Pa· s)/m]
ρ_0 = equilibrium (ambient) density [kg/m^3]
$c = \frac{dx}{dt}$ is the **phase speed** (speed of sound) [m/s]
k = wave number or propagation constant [rad./m]
r = radial distance from the center of the sphere [m]

P ACOUSTIC POWER, SPHERICAL WAVES [W]

For a constant acoustic power, intensity increases proportional to a reduction in dispersion area.

$$\Pi = \int_S \vec{I} \cdot d\vec{s} \quad \text{general definition}$$

$$\Pi = \underbrace{4\pi r^2}_{\text{area of a spherical surface}} I \quad \text{for spherical dispersion}$$

$$\Pi = \underbrace{2\pi r^2}_{\text{hemi-spherical surface}} I \quad \text{for hemispherical dispersion}$$

S = surface surrounding the sound source, or at least the surface area through which all of the sound passes [m^2]
I = acoustic intensity [W/m^2]
r = radial distance from the center of the sphere [m]

FREQUENCY BANDS

f_u f_l FREQUENCY BANDS

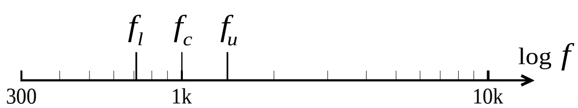
The human ear perceives different frequencies at different levels. Frequencies around 3000 Hz appear loudest with a rolloff for higher and lower frequencies. Therefore in the analysis of sound levels, it is necessary to divide the frequency spectrum into segments or bands.

$$\boxed{f_u = 2^{1/N} f_l} \quad \text{a } \frac{1}{N}\text{-octave band}$$

f_u = the upper frequency in the band [Hz]
f_l = the lowest frequency in the band [Hz]
N = the bandwidth in terms of the (inverse) fractional portion of an octave, e.g. N=2 describes a ½-octave band

f_c CENTER FREQUENCY [Hz]

The center frequency is the geometric mean of a frequency band.

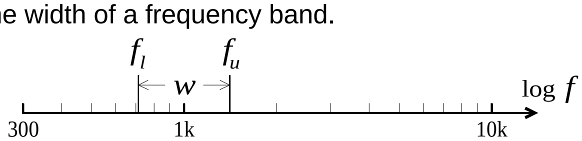


$$\boxed{f_c = \sqrt{f_u f_l}}$$

f_u = the upper frequency in the band [Hz]
f_l = the lowest frequency in the band [Hz]

w BANDWIDTH [Hz]

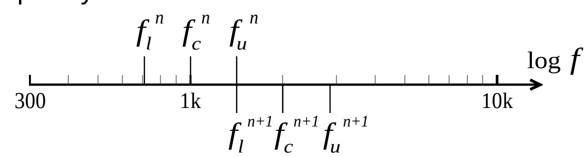
The width of a frequency band.



$$\boxed{w = f_u - f_l = \left(2^{\frac{1}{2N}} - 2^{-\frac{1}{2N}}\right) f_c}$$

CONTIGUOUS BANDS

The upper frequency of one band is the lower frequency of the next.



$$\boxed{\frac{f_c^{\,n+1}}{f_c^{\,n}} = 2^{\frac{1}{N}}}$$

Octave bands are the most common contiguous bands:

$$\frac{f_c^{\,n+1}}{f_c^{\,n}} = 2 \qquad f_l = \frac{f_c}{\sqrt{2}} \qquad f_u = f_c\sqrt{2} \qquad w = \frac{f_c}{\sqrt{2}}$$

e.g. for f_c = 1000 Hz, f_l = 707 Hz, f_u = 1414 Hz

STANDARD CENTER FREQUENCIES [Hz]

Octave bands:
16, 31.5, 63, 125, 250, 500, 1000, 2000, 4000, 8000

1/3-Octave bands:
10, 12.5, 16, 20, 25, 31.5, 40, 50, 63, 80, 100, 125, 160, 200, 250, 315, 400, 500, 630, 800, 1000, 1250, 1600, 2000, 2500, 3150, 4000, 5000, 6300, 8000

MUSICAL INTERVALS [Hz]

Each half-step is $2^{1/12}$ times higher than the previous note.

Harmonious frequency ratios:

2:1	octave	$2^{12/12}$ = 2.000	2/1 = 2.000
3:2	perfect fifth	$2^{7/12}$ = 1.489	3/2 = 1.500
4:3	perfect fourth	$2^{5/16}$ = 1.335	4/3 = 1.333
5:4	major third	$2^{4/12}$ = 1.260	5/4 = 1.200

IL INTENSITY LEVEL [dB]

Acoustic intensity in decibels. Note that $IL = SPL$ when IL is referenced to 10^{-12} and SPL is referenced to 20×10^{-6}.

Intensity Level: $$\boxed{IL = 10\log\left(\frac{I}{I_{\text{ref}}}\right)}$$

I = acoustic intensity [W/m^2]
I_{ref} = the reference intensity 1×10^{-12} in air [W/m^2]

I_f SPECTRAL FREQUENCY DENSITY [W/m^2]

The distribution of acoustic intensity over the frequency spectrum; the intensity at frequency f over a bandwidth of Δf. The bandwidth Δf is normally taken to be 1 Hz and may be suppressed.

$$I_f(f) = \frac{\Delta I}{\Delta f}$$

ISL INTENSITY SPECTRUM LEVEL [dB]

The spectral frequency density expressed in decibels. This is what you see on a spectrum analyzer.

$$ISL(f) = 10\log\frac{I_f(f)\cdot 1\text{Hz}}{I_\text{ref}}$$

I_ref = the reference intensity 10^{-12} [Pa]

IL_BAND FREQUENCY BAND INTENSITY LEVEL [dB]

The sound intensity within a frequency band.

$$IL_\text{BAND} = ISL + 10\log w$$

ISL = intensity spectrum level [dB]
w = bandwidth [Hz]

SPL SOUND PRESSURE LEVEL [dB]

Acoustic pressure in decibels. Note that $IL = SPL$ when IL is referenced to 10^{-12} and SPL is referenced to 20×10^{-6}. An increase of 6 dB is equivalent to doubling the amplitude. A spherical source against a planar surface has a 3 dB advantage over a source in free space, 6 dB if it's in a corner, 9 dB in a 3-wall corner.

Sound Pressure Level: $\boxed{SPL = 20\log\left(\frac{P_e}{P_\text{ref}}\right)}$

for Multiple Sources:

$$\boxed{SPL = 10\log\left[\sum_{i=1}^{N}\left(\frac{P_{ei}}{P_\text{ref}}\right)^2\right] = 10\log\left(\sum_{i=1}^{N}10^{SPL_i/10}\right)}$$

for Multiple Identical Sources:

$$\boxed{SPL = SPL_0 + 10\log N}$$

P_e = effective or rms acoustic pressure [Pa]
P_ref = the reference pressure 20×10^{-6} in air, 1×10^{-6} in water [Pa]
N = the number of sources

PSL PRESSURE SPECTRUM LEVEL [dB]

Same as intensity spectrum level.

$$PSL(f) = ISL(f) = SPL \text{ in a 1 Hz band}$$

SPL = sound pressure level [dB]

dBA WEIGHTED SOUND LEVELS (13.2)

Since the ear doesn't perceive sound pressure levels uniformly across the frequency spectrum, several correction schemes have been devised to produce a more realistic scale. The most common is the A-weighted scale with units of dBA. From a reference point of 1000 Hz, this scale rolls off strongly for lower frequencies, has a modest gain in the 2-4 kHz region and rolls off slightly at very high frequencies. Other scales are dBB and dBC. Most standards, regulations and inexpensive sound level meters employ the A-weighted scale.

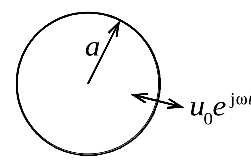
ACOUSTICAL SOURCES

MONOPOLE (7.1)

The monopole source is a basic theoretical acoustic source consisting of a small (small ka) pulsating sphere.

$$\boxed{p(r,t) = \frac{A}{r}e^{j(\omega t - kr)}}$$

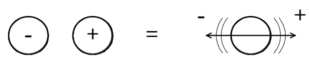
where $A = jka^2\rho_0 c u_0$

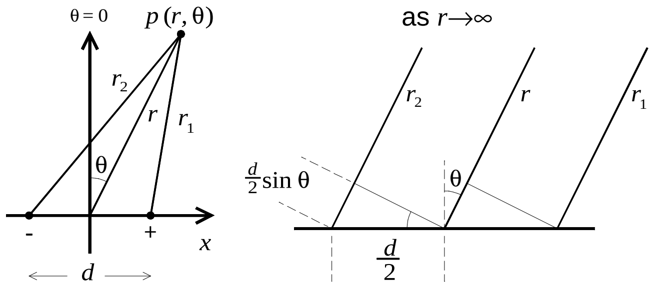


$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]
r = radial distance from the center of the source [m]
ω = frequency [rad/s]
k = wave number or propagation constant [rad./m]
ρ_0 = equilibrium (ambient) density [kg/m^3]
$c = \frac{dx}{dt}$ is the phase speed (speed of sound) [m/s]
u = particle velocity (due to oscillation, not flow) [m/s]

DIPOLE

The dipole source is a basic theoretical acoustic source consisting of two adjacent monopoles 180° out of phase. Mathematically this approximates a single source in translational vibration, which is what we really want to model.





$$p(r,\theta) = \frac{A}{r_1}e^{\mathrm{j}(\omega t - kr_1)} - \frac{A}{r_2}e^{\mathrm{j}(\omega t - kr_2)}$$

where $A = \mathrm{j}ka^2\rho_0 c u_0$

Far from the source, the wave looks spherical:

$$\boxed{p(r,\theta) = \mathrm{j}2\underbrace{\frac{A}{r}e^{\mathrm{j}(\omega t - kr)}}_{\text{spherical wave}}\underbrace{\sin\left(\tfrac{1}{2}kd\sin\theta\right)}_{\text{directivity function}}}$$

$p = \mathscr{P} - \mathscr{P}_0$ acoustic pressure [Pa]

r = radial distance from the center of the source [m]

ω = frequency [rad/s]

k = wave number or propagation constant [rad./m]

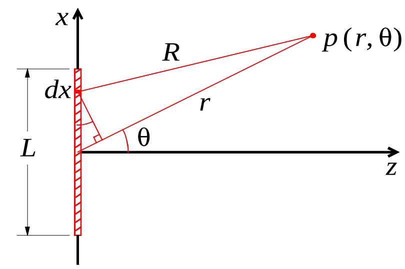
ρ_0 = equilibrium (ambient) density [kg/m^3]

$c = \frac{dx}{dt}$ is the phase speed (speed of sound) [m/s]

u = particle velocity (due to oscillation, not flow) [m/s]

LINE SOURCE (7.3)

A line source of length L is calculated as follows.



Let $P_{dx}(R,\theta,t) = \frac{dx}{L}\frac{A}{R}e^{\mathrm{j}(\omega t - kR)}$

where P_{dx} is the pressure at a remote point due to one tiny segment of the line source,

and $A = \mathrm{j}ka^2\rho_0 c u_0$.

for $r \gg L$, $R \approx r - x\sin\theta$,

$p(r,\theta) = \int_L P_{dx}(r,\theta,t)dx$ (abbreviated form)

$$P_{dx}(r,\theta,t) \approx \frac{dx}{L}\frac{A}{r}e^{\mathrm{j}(\omega t - kr + kx\sin\theta)}$$

$$p(r,\theta,t) = \frac{1}{L}\frac{A}{r}e^{\mathrm{j}(\omega t - kr)}\int_{-L/2}^{L/2} e^{\mathrm{j}kx\sin\theta}dx$$

$$\boxed{p(r,\theta,t) = \frac{A}{r}e^{\mathrm{j}(\omega t - kr)}D(\theta)}$$ where

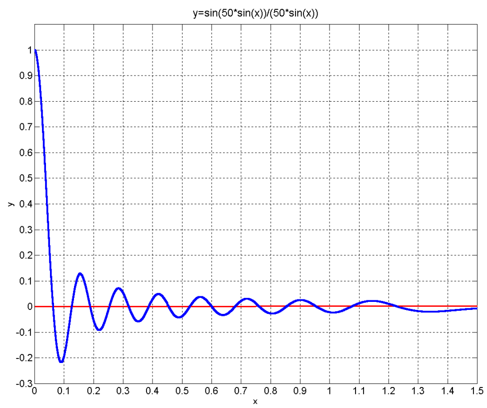
$$\boxed{D(\theta) = \frac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta}}$$ **Directivity Function**

see also Half Power Beamwidth p16.

DIRECTIVITY FUNCTION

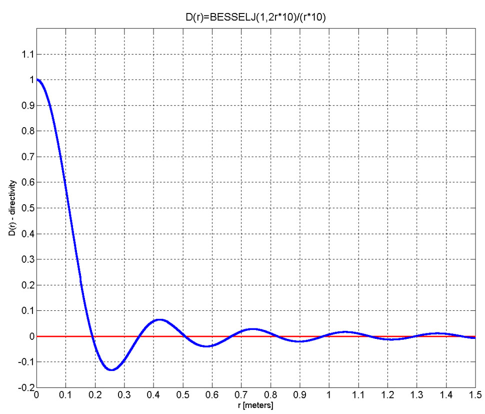
The directivity function is responsible for the lobes of the dispersion pattern. The function is normalized to have a maximum value of 1 at θ = 0. Different directivity functions are used for different elements; the following is the directivity function for the **line source**.

$$D(\theta)=\frac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta}$$



The following is the directivity function for a **focused source**.

$$D(r)=\frac{J_1(2Gr/a)}{Gr/a}$$



$J_1(x)$ = first order Bessel J function

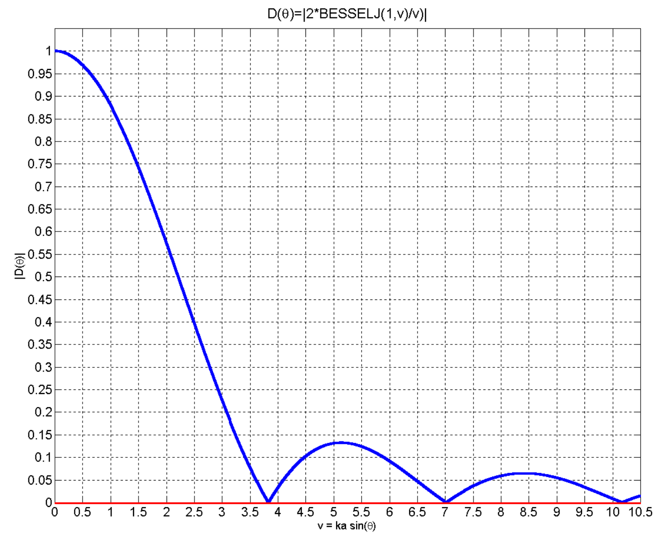
see also Half Power Beamwidth p16.

CIRCULAR SOURCE (7.4, 7.5a)

A speaker in an enclosure may be modeled as a circular source of radius a in a rigid infinite baffle vibrating with velocity $\mu_0 e^{j\omega t}$. For the far field pressure ($r > \frac{1}{2}ka^2$).

$$\boxed{p(r,\theta,t)=\mathrm{j}\frac{ka^2}{2r}\rho_0 c\mu_0 D(\theta)e^{\mathrm{j}(\omega t-kr)}}$$

where $D(\theta)=\dfrac{2J_1(ka\sin\theta)}{ka\sin\theta}$ **Directivity function**

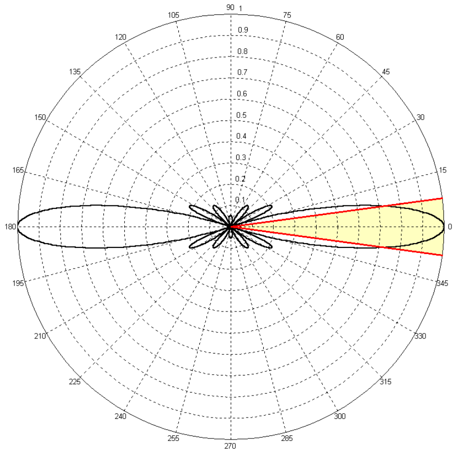


$$\theta_{\text{null }1}\approx\sin^{-1}\left(\frac{3.83}{ka}\right)$$ **First null**

$J_1(x)$ = first order Bessel J function
r = radial distance from the source [m]
θ = angle with the normal from the circular source [radians]
t = time [s]
$\rho_0 c$ = impedance of the medium [rayls] (415 for air)
k = wave number or propagation constant [rad./m]

$2\theta_{HP}$ HALF-POWER BEAMWIDTH

The angular width of the main lobe to the points where power drops off by 1/2; this is the point at which the directivity function equals $1/\sqrt{2}$.



For a circular source:

from the Directivity function: $D(\theta_{HP}) = \dfrac{1}{\sqrt{2}} = \dfrac{2J_1(ka\sin\theta)}{ka\sin\theta}$

$$ka\sin\theta_{HP} = 1.61634 \quad\Rightarrow\quad 2\theta_{HP} = 2\sin^{-1}\left(\frac{1.61634}{ka}\right)$$

$$2\theta_{HP} \approx \frac{3.2327}{ka}\text{radians} \quad \text{or} \quad \frac{185.22}{ka}\text{degrees,} \quad \text{for } ka \gg 1$$

$J_1(x)$ = first order Bessel J function

For a line source:

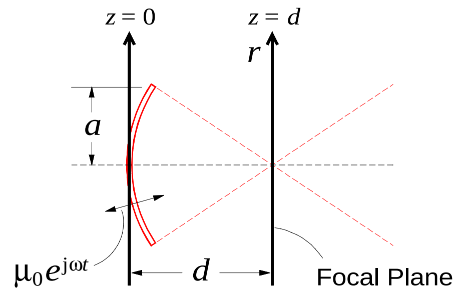
from the Directivity function: $D(\theta_{HP}) = \dfrac{1}{\sqrt{2}} = \dfrac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta}$

$$\tfrac{1}{2}kL\sin\theta_{HP} = 1.391558 \quad\Rightarrow\quad 2\theta_{HP} = 2\sin^{-1}\left(\frac{1.391558}{\frac{1}{2}kL}\right)$$

FOCUSED SOURCE

The dispersion pattern of a focused source is measured at the **focal plane**, a plane passing through the focal point and perpendicular to the central axis.

FOCUSED SOURCE



Focal Plane pressure $\boxed{|p(r)| = G\rho_0 c|D(r)|}$

where $G = \dfrac{ka^2}{2d}$

and $D(r) = \dfrac{J_1(2Gr/a)}{Gr/a}$ (Directivity function)

$J_1(x)$ = first order Bessel J function
r = radial distance from the central axis [m]
G = constant [radians]
a = radius of the source [m]
d = focal length [m]
$\rho_0 c$ = impedance of the medium [rayls] (415 for air)
k = wave number or propagation constant [rad./m]

z_0 RAYLEIGH NUMBER [rad. · m]

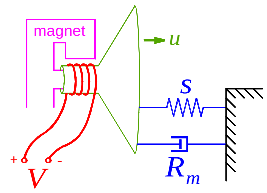
The Rayleigh number or Rayleigh length is the distance along the central axis from a circular piston element to the beginning of the **far field**. Beyond this point, complicated pressure patterns of the near field can be ignored.

$$\boxed{z_0 = \frac{\pi a^2}{\lambda} = \tfrac{1}{2}ka^2}$$

a = radius of the source [m]
d = focal length [m]
$\rho_0 c$ = impedance of the medium [rayls] (415 for air)
k = wave number or propagation constant [rad./m]

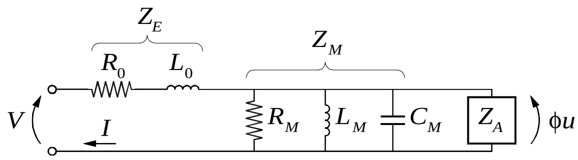
MOVING COIL SPEAKER (14.3b, 14.5)

Model for the moving coil loudspeaker.



$V = \phi u$ Faraday's law

$F = \phi I$ Lorentz force



u = velocity of the voice coil [m/s]

I = electrical current [A]

R_0 = electrical resistance of the voice coil [Ω]

L_0 = electrical inductance of the voice coil [H]

s = spring stiffness due to flexible cone suspension material [N/m]

R_m = mechanical resistance, a small frictional force [(N· s)/m or kg/s]

R_M = effective electrical resistance due to the mechanical resistance of the system [Ω]

C_M = effective electrical capacitance due to the mechanical stiffness [F]

L_M = effective electrical inductance due to the mechanical inertia [H]

V = voltage applied to the voice coil [V]

Z_E = electrical impedance due to electrical components [Ω]

Z_A = effective electrical impedance due to mechanical air loading [Ω]

Z_M = effective electrical impedance due to the mechanical effects of spring stiffness, mass, and (mechanical) resistance [Ω]

F = force on the voice coil [V]

$\phi = Bl$ coupling coefficient [N/A]

B = magnetic field [Tesla (an SI unit)]

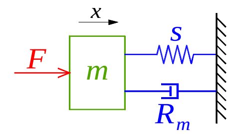
l = length of wire in the voice coil [m]

Z_m MECHANICAL IMPEDANCE $[(\mathrm{N\cdot s})/\mathrm{m}]$

(1.7)

The mechanical impedance is analogous to electrical impedance but does not have the same units. Where electrical impedance is voltage divided by current, mechanical impedance is force divided by speed, sometimes called *mechanical ohms*.

$$\boxed{Z_m = \frac{F}{u}}$$



$$F \underbrace{-sx}_{\substack{\text{Return}\\\text{force of}\\\text{the spring}}} - \underbrace{Rm\dot{x}}_{\substack{\text{Force due to}\\\text{mechanical}\\\text{resistance}}} = \underbrace{m\ddot{x}}_{\substack{\text{Mass}\times\\\text{acceleration}}}$$

$$F = m\ddot{x} + R_m\dot{x} + sx$$

let $F(t) = \tilde{F}(\omega)e^{j\omega t}$ and $X(t) = \tilde{X}(\omega)e^{j\omega t}$

then $\tilde{F}\,\cancel{e^{j\omega t}} = \left(-\omega^2 m + j\omega R_m + s\right)\tilde{X}\,\cancel{e^{j\omega t}}$

and $\tilde{X} = \dfrac{\tilde{F}}{-\omega^2 m + j\omega R_m + s}$

so the velocity $\tilde{U} = \frac{dx}{dt} = j\omega\tilde{X} = \dfrac{j\omega\tilde{F}}{-\omega^2 m + j\omega R_m + s}$

finally

$$Z_{mo} = \frac{\tilde{F}}{\tilde{U}} = \frac{\tilde{F}}{j\omega\tilde{F}/\left(-\omega^2 m + j\omega R_m + s\right)} = \frac{-\omega^2 m + j\omega R_m + s}{j\omega}$$

$$\boxed{Z_{mo} = \underbrace{R_m}_{\text{damping}} + \underbrace{j\omega m}_{\substack{\text{inertia}\\\text{due to}\\\text{mass}}} - \underbrace{j\frac{s}{\omega}}_{\substack{\text{spring}\\\text{effect}}}}$$

m = mass of the speaker cone and voice coil [kg]

x = distance in the direction of motion [m]

s = spring stiffness due to flexible cone suspension material [N/m]

R_m = mechanical resistance, a small frictional force $[(\mathrm{N\cdot s})/\mathrm{m}$ or kg/s]

F = force on the speaker mass [N]

ω = frequency in radians

Z_M ELECTRICAL IMPEDANCE DUE TO MECHANICAL FORCES [Ω]

Converting mechanical impedance to electrical inverts each element.

$$R_M = \frac{\phi^2}{R_{mo}}, \quad C_M = \frac{m}{\phi^2}, \quad L_M = \frac{\phi^2}{s}$$

$$\boxed{Z_M = \frac{\phi^2}{Z_{mo}} = R_M \parallel C_M \parallel L_M}$$

$$Z_{mo} = R_m + \mathrm{j}\omega m - \mathrm{j}\frac{s}{\omega} \quad \rightarrow \quad Z_M = \frac{\phi^2}{R_m + \mathrm{j}\omega m - \mathrm{j}\frac{s}{\omega}}$$

R_M = effective electrical resistance due to the mechanical resistance of the system [Ω]
C_M = effective electrical capacitance due to the mechanical stiffness [F]
L_M = effective electrical inductance due to the mechanical inertia [H]
R_m = mechanical resistance, a small frictional force [(N·s)/m or kg/s]
m = mass of the speaker cone and voice coil [kg]
s = spring stiffness due to flexible cone suspension material [N/m]
$\phi = Bl$ coupling coefficient [N/A]
Z_{mo} = mechanical impedance, open-circuit condition [(N·s)/m or kg/s]

Z_A ELECTRICAL IMPEDANCE DUE TO AIR [Ω]

The factor of two in the denominator is due to loading on both sides of the speaker cone.

$$\boxed{Z_A = \frac{\phi^2}{2Z_r}}$$

Z_r RADIATION IMPEDANCE [(N·s)/m] (7.5)

This is the mechanical impedance due to air resistance. For a circular piston:

$$\boxed{Z_r = \rho_0 c S\left[R_1(2ka) + \mathrm{j}X_1(2ka)\right]}$$

$$Z_r \approx \begin{cases} \mathrm{j}\frac{8}{3}\rho_0 a^3\omega, & ka \ll 1 \\ \pi a^2 \rho_0 c, & ka \gg 1 \end{cases}$$

The functions R_1 and X_1 are defined as:

$$R_1(x) = 1 - \frac{2J_1(x)}{x} \simeq \frac{x^2}{8} - \frac{x^4}{192} + \frac{x^6}{9216} - \frac{x^8}{737280}$$

$$X_1(x) = \frac{2\mathrm{H}_1(x)}{x} \simeq \begin{cases} \frac{4}{\pi}\left(\frac{x}{3} - \frac{x^3}{45} + \frac{x^5}{1600} - \frac{x^7}{10^5} + \frac{x^9}{10^7}\right), & x \le 4.32 \\ \frac{4}{\pi x} + \sqrt{\frac{8}{\pi x^3}}\sin\left(x - \frac{3\pi}{4}\right), & x > 4.32 \end{cases}$$

$\rho_0 c$ = impedance of the medium [rayls] (415 for air)
S = surface area of the piston [m^2]
J_1 = first order Bessel J function
R_1 = a function describing the real part of Z_r
X_1 = a function describing the imaginary part of Z_r
x = just a placeholder here for $2ka$
k = wave number or propagation constant [rad./m]
a = radius of the source [m]
H_1 = first order Struve function
ω = frequency in radians

m_r RADIATION MASS [kg] (7.5)

The effective increase in mass due to the loading of the fluid (radiation impedance).

$$\boxed{m_r = \frac{X_r}{\omega}}$$

The effect of radiation mass is small for light fluids such as air but in a more dense fluid such as water, it can significantly decrease the resonant frequency.

$$\omega_0 = \sqrt{\frac{s}{m}} \quad \rightarrow \quad \sqrt{\frac{s}{m + m_r}}$$

The functions R_1 and X_1 are defined as:

X_r = radiation reactance, the imaginary part of the radiation impedance [(N·s)/m]
ω = frequency in radians
s = spring stiffness due to flexible cone suspension material [N/m]
m = mass of the speaker cone and voice coil [kg]

p_{axial} **AXIAL PRESSURE** [Pa] (7.4a)

$$p_{\text{axial}} = \mathrm{j}\frac{ka^2}{2r}\rho_0 c\,u = \mathrm{j}\frac{\omega\rho_0 a^2}{2r}u, \quad r > \tfrac{1}{2}ka^2$$

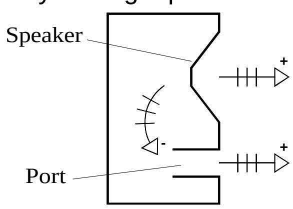
$$u = \frac{Z_{MA}}{Z_{MA}+Z_E}\frac{V}{\phi}$$

$$\boxed{p_{\text{axial}} = \frac{Z_{MA}}{Z_{MA}+Z_E}\frac{\mathrm{j}\omega\rho_0 a^2}{2\phi r}V_0\,e^{\mathrm{j}\omega t}}$$

$Z_{MA} = Z_M \parallel Z_A$ = effective electrical impedance due to the mechanical components and the effect of air [Ω]
Z_E = electrical impedance due to the voice coil [Ω]

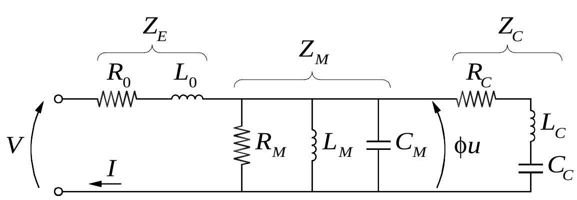
BASS REFLEX ENCLOSURE (14.6c)

The bass response of a speaker/cabinet system can be improved bat the expense of an increase in low frequency rolloff by adding a port to the enclosure.



Choose ω_c somewhat less than $\sqrt{s/m}$ to add a response peak just below the existing damping-controlled peak. Rolloff below that point will increase from 12 dB/octave to 18 dB/octave.

$$\omega_c = \frac{1}{\sqrt{L_c C_c}}, \quad L_c = \frac{\phi^2}{s_c}, \quad C_c = \frac{m_v}{\phi^2}$$



L_c = effective electrical inductance due to the cabinet [H]
s_c = cabinet stiffness [N/m]
C_c = effective electrical capacitance due to the cabinet [F]
m_v = mass of the air inside the port or vent [kg]
$\phi = Bl$ coupling coefficient [N/A]

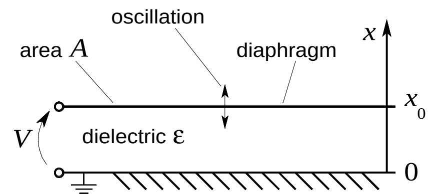
PIEZOELECTRIC TRANSDUCER (14.12b)

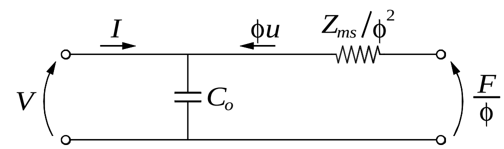
Uses a crystal (usually quartz) or a ceramic; voltage is proportional to strain. High efficiency (30% is high for acoustics.) Highly resonant. Used for microphones and speakers.

ELECTROSTATIC TRANSDUCER

(14.3a, 14.9)

A moving diaphragm of area A is separated from a stationary plate by a dielectric material (air). A bias voltage is applied between the diaphragm and plate. Modern devices use a PVDF film for the diaphragm which has a permanent charge, so no bias voltage is required. **Bias**, in this case and in general, is an attempt to linearize the output by shifting its operating range to a less non-linear operating region. The DC bias voltage is much greater in magnitude than the time-variant signal voltage but is easily filtered out in signal processing.





Acoustic voltage: $\boxed{V = \frac{1}{\mathrm{j}\omega C_0}(I+\phi u)}$,

Mechanical voltage: $\frac{F}{\phi} = \frac{1}{\mathrm{j}\omega C_o}I + \frac{Z_{ms}}{\phi^2}\phi u$,

For this circuit model, there is no inverting of mechanical impedances as in the loudspeaker circuit.

Coupling coefficient: $\phi = \frac{C_0 V_0}{x_0}$,

Equilibrium capacitance: $C_0 = \frac{\varepsilon A}{x_0}$

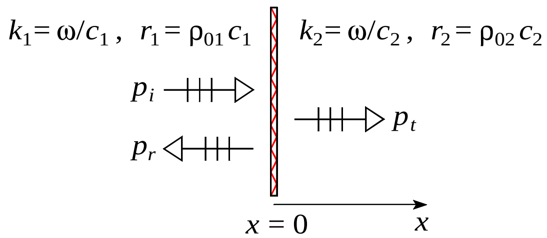
V_0 = bias voltage [V]
C_0 = equilibrium capacitance due to diaphragm, back plate, and dielectric [F]
F = force on the diaphragm [N]
I = electrical current [A]
ϕu = electrical current due to mechanical force [A]
ϕ = coupling coefficient [N/V]
u = acoustic velocity [m/s]
Z_{ms} = short-circuit mechanical impedance [(N· s)/m]
x_0 = equilibrium position of the diaphragm [m]
A = area of the diaphragm [m^2]

REFLECTION AND TRANSMISSION AT NORMAL INCIDENCE (6.2)

Incident: $p_i = P_i e^{j(\omega t - k_1 x)}$

Reflected: $p_r = P_r e^{j(\omega t + k_1 x)}$

Transmitted: $p_t = P_t e^{j(\omega t - k_2 x)}$



Boundary Conditions:

1) Pressure is equal across the boundary at x=0.

$$p_i + p_r = p_t \rightarrow P_i + P_r = P_t$$

2) Continuity of the normal component of velocity.

$$u_i + u_r = u_t$$

R, T, R_I, T_I REFLECTION AND TRANSMISSION COEFFICIENTS (6.2)

The ratio of reflected and transmitted magnitudes to incident magnitudes. The stiffness of the medium has the most effect on reflection and transmission.

i) $r_2 >> r_1$ Medium 2 is very hard compared to medium 1 and we have total reflection. $R \approx 1,\ T \approx 2$

Note that T=2 means that the amplitude doubles, but there is practically no energy transmitted due to high impedance.

ii) $r_2 = r_1$ The mediums are similar and we have total transmission. $R = 0,\ T = 1$

iii) $r_2 << r_1$ Medium 2 is very soft and we have total reflection with the waveform inverted. $R \approx -1,\ T \approx 0$

$$R = \frac{P_r}{P_i} = \frac{r_2 - r_1}{r_2 + r_1} \qquad R_I = \left(\frac{r_2 - r_1}{r_2 + r_1} \right)^2$$

$$T = \frac{P_t}{P_i} = \frac{2r_2}{r_2 + r_1} \qquad T_I = \frac{4 r_2 r_1}{\left(r_2 + r_1 \right)^2}$$

P_i, P_r, P_t = peak acoustic pressure (or magnitude) of incident, reflected, and transmitted waves [Pa]

r_1, r_2 = characteristic acoustic impedances of the materials $(\rho_0 c)_1$, $(\rho_0 c)_2$ [rayls or (Pa · s)/m]

ρ_0 = equilibrium (ambient) density [kg/m^3]

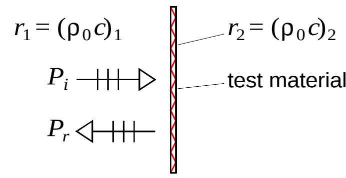
c = the **phase speed** (speed of sound, 343 m/s in air) [m/s]

R_I = reflection intensity coefficient [no units]

T_I = transmission intensity coefficient [no units]

USING REFLECTION TO DETERMINE MATERIAL PROPERTIES (6.1)

The impedance of a material (and thereby its density) can be determined by bouncing a plane wave off of the material at normal incidence and measuring the relative sound pressure levels. However, there are two possible results since we don't know the phase of the reflected wave, i.e. P_r can be positive or negative.



$$\text{SPL}_{\text{difference}} = 20 \log \frac{P_i}{\pm P_r} \quad \text{and} \quad \frac{\pm P_r}{P_i} = R = \frac{r_2 - r_1}{r_2 + r_1}$$

P_i = peak acoustic pressure, incident [Pa]

P_r = peak acoustic pressure, reflected [Pa]

r_1 = characteristic acoustic impedances of the known material $(\rho_0 c)_1$ [rayls or (Pa · s)/m]

r_2 = characteristic acoustic impedance of the unknown material $(\rho_0 c)_2$ [rayls or (Pa · s)/m]

ρ_0 = equilibrium (ambient) density [kg/m^3]

c = the **phase speed** (speed of sound, 343 m/s in air) [m/s]

a ABSORPTION COEFFICIENT

The absorption coefficient can be measured in an impedance tube by placing a sample at the end of the tube, directing an acoustic wave onto it and measuring the standing wave ratio.

$$\boxed{a = \frac{W_a}{W_i} = 1 - \frac{SWR-1}{SWR+1}}$$

An alternative method is to place a sample in a reverberation room and measuring the effect on the reverberation time for the room. Difficulties with this method include variations encountered due to the location of the sample in the room and the presence of standing waves at various frequencies.

$$\boxed{a_s = a_0 + \frac{0.161V}{S_s}\left(\frac{1}{T_s} - \frac{1}{T_0}\right)}$$

W_i = power incident on a surface [W]
W_a = power absorbed by ? [W]
W_r = W_i - W_a = power in the reflected sound [W]
a_s = absorption coefficient of the sample [no units]
a_0 = absorption coefficient of the empty room [no units]
V = volume of the room [m^3]
S_s = surface area of the sample [no units]
T_s = reverberation time with the same in place [s]
T_0 = reverberation time in the empty room [s]

a Absorption Coefficient, Selected Materials [no units]

	250 Hz	1 kHz	4 kHz
acoustic tile suspended ceiling	0.50	0.75	0.60
brick	0.03	0.04	0.07
carpet	0.06	0.35	0.65
concrete	0.01	0.02	0.02
concrete block, painted	0.05	0.07	0.08
fiberglass, 1" on rigid backing	0.25	0.75	0.65
glass, heavy plate	0.06	0.03	0.02
glass, windowpane	0.25	0.12	0.04
gypsum, ½' on studs	0.10	0.04	0.09
floor, wooden	0.11	0.07	0.07
floor, linoleum on concrete	0.03	0.03	0.02
floor, terrazzo	0.01	0.02	0.02
upholstered seats	0.35	0.65	0.60
wood paneling, 3/8-1/2"	0.25	0.17	0.10

TRANSMISSION THROUGH PARTITIONS

TL TRANSMISSION LOSS THROUGH A THIN PARTITION [dB] (13.15a)

For a planar, nonporous, homogeneous, flexible wall, the transmission loss is dependent on the density of the partition and the frequency of the noise.

$$P_i + P_r = \left(1 + \mathrm{j}\frac{\omega\rho_s}{\rho_0 c}\right)P_t \quad T_I = \left(\frac{2\rho_0 c}{\omega\rho_s}\right)^2$$

$$\boxed{TL = 20\log\frac{I_i}{I_t} = 20\log\left(f\rho_s\right) - 20\log\frac{\rho_0 c}{\pi}}$$

This is the **Normal Incidence Mass Law**. Low frequency roll off of the transmitted wave will be 6 dB/octave. Doubling the mass of the wall will give an additional 6 dB loss.

Loss through a thin partition in air ($\rho_0 c$ = 415)**:**

$$TL_0 = 20\log\frac{I_i}{I_t} = 20\log\left(f\rho_s\right) - 42$$

Transmission loss as a function of power:

$$TL_0 = 10\log\frac{W_i}{W_t}$$

ρ_s = surface density of the partition material [kg/m^2]
$\rho_0 c$ = impedance of the medium [rayls or (Pa·s)/m] (415 for air)
f = frequency [Hz]
P_i = peak acoustic pressure, incident [Pa]
P_r = peak acoustic pressure, reflected [Pa]
P_t = peak acoustic pressure, transmitted [Pa]
I_i = intensity of the incident wave [W/m^2]
I_t = intensity of the transmitted wave [W/m^2]
W_i = power of the incident wave [W/m^2]
W_t = power of the transmitted wave [W/m^2]

ρ_s SURFACE DENSITY [kg/m^2]

The surface density affects the transmission loss through a material and is related to the material density.

$$\boxed{\rho_s = \rho_0 h}$$

ρ_0 = density of the material [kg/m^3]
h = thickness of the material [m]

TL TRANSMISSION LOSS IN COMPOSITE WALLS AT NORMAL INCIDENCE [dB]

For walls constructed of multiple materials, e.g. a brick wall having windows, the transmission loss is the sum of the transmission losses in the different materials.

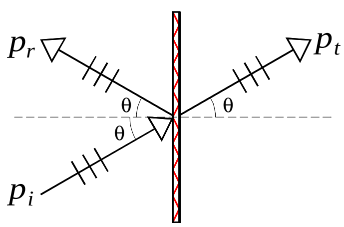
$$TL_0 = 10\log\frac{1}{\bar{\bar{T}}_I}\ ,\ \textit{where}\ \bar{\bar{T}}_I = \frac{1}{S}\sum_i T_i S_i\ ,$$

for a wall in air: $T_i = \left(\dfrac{132}{f\rho_S}\right)^2$

S_i = area of the i^{th} element [m^2]
T_i = T_I for of the i^{th} element (transmission intensity coefficient [no units]
ρ_s = surface density [kg/m^2]

TRANSMISSION AT OBLIQUE INCIDENCE [dB]

Waves striking a wall at an angle see less impedance than waves at normal incidence.



$$T_I(\theta) = \frac{1}{1+\left(\dfrac{\omega\rho_S}{2\rho_0 c}\cos\theta\right)^2}$$

T_I = transmission intensity coefficient [no units]
θ = angle of incidence [radians]
$\rho_0 c$ = impedance of the medium [rayls or (Pa · s)/m] (415 for air)
ρ_s = surface density [kg/m^2]

DIFFUSE FIELD MASS LAW [dB]

In a diffuse field, sound is incident by definition at all angles with equal probability. Averaging yields an increase in sound transmission of 5 dB over waves of normal incidence.

$$TL_{\text{diffuse}} = TL_0 - 5$$

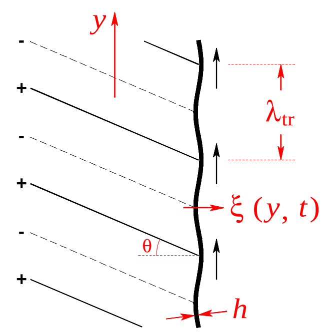
Loss through a thin partition in air ($\rho_0 c$ = 415)**:**

$$TL_{\text{diffuse}} = 20\log(f\rho_S) - 47$$

COINCIDENCE EFFECT (13.15a)

When a plane wave strikes a thin partition at an angle, there are alternating high and low pressure zones along the partition that cause it to flex sinusoidally. This **flexural wave** propagates along the surface of the wall. At some frequency, there is a kind of resonance and the wall becomes transparent to the wave. This causes a marked decrease in the transmission loss over what is expected from the mass law; it can be 10-15 dB.

Coincidence occurs when $\lambda_{tr} = \lambda_p$.



The wave equation for a thin plate:

$$\frac{\partial^4\xi}{\partial y^4} + \frac{12}{h^2 c_{\text{bar}}^2}\frac{\partial^2\xi}{\partial t^2} = 0$$

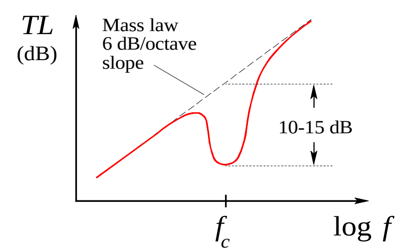
Particle displacement: $\xi(y,t) = e^{j\omega(t - y/C_p)}$

Dispersion: $C_p(f) = \sqrt{\dfrac{\pi}{\sqrt{3}} h c_{\text{bar}} f}$ [m/s]

Trace wavelength: $\lambda_{tr} = \dfrac{\lambda}{\sin\theta}$ [m]

Flexural wavelength: $\lambda_p = \dfrac{C_p}{f}$ [m]

Coincidence frequency: $f_c = \dfrac{c^2}{1.8 h c_{\text{bar}}}$ [Hz]



Design considerations: If $f < f_c$, use the diffuse field mass law to find the transmission loss. If $f > f_c$, redesign to avoid. Note that f_c is proportional to the inverse of the thickness.

ξ = transverse particle displacement [m]
h = panel thickness [m]
c_{bar} = bar speed for the panel material [m/s]
t = time [s]
θ = angle of incidence [radians]

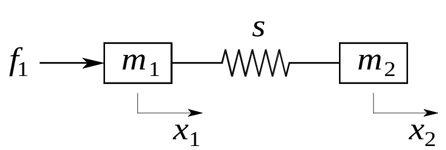
DOUBLE WALLS

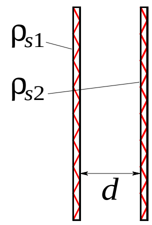
Masses in series look like series electrical connections. We want to determine the motion of the second wall due to sound incident on the first. Assume that $d << \lambda$ and let:

$m_i = \rho_{Si}$ mass per unit area of i^{th} wall

$s = \dfrac{\gamma P_0}{d} = \dfrac{\rho_0 c}{d}$ stiffness per unit area of air

$f_1 = p_i + p_r$ force per unit area on wall 1





From **Newton's Law** $F=ma$:

Mass 1: $f_1 - s\left(x_1 - x_2\right) = m_1 \ddot{x}_1$

Mass 2: $s\left(x_1 - x_2\right) = m_2 \ddot{x}_2$

Let $f_1 = F_1 e^{j\omega t}$, $x_i = X_i(\omega)\, e^{j\omega t}$

$$\overbrace{\begin{bmatrix} s - m_1\omega^2 & -s \\ -s & s - m_2\omega^2 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} F_1 \\ 0 \end{bmatrix}}^{b}$$

Apply **Cramer's rule**, $X_i = \dfrac{\Delta_i}{\Delta}$

where $\Delta = \det A$ and

$\Delta_i = \det A$, with b in the i^{th} column

The **wall impedance** is

$$Z_W = \frac{F_1}{j\omega}\frac{\Delta}{\Delta_2} = \frac{\Delta}{j\omega s} = j\left[\left(m_1 + m_2\right)\omega - \frac{m_1 m_2}{s}\omega^3\right]$$

$$\to Z_W = j\left[\left(\rho_{S1} + \rho_{S2}\right)\omega - \frac{\rho_{S1}\rho_{S2} d}{\rho_0 c^2}\omega^3\right]$$

Resonance occurs at Z_W=0:

$$\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{\rho_0 c^2}{d}\left(\frac{1}{\rho_{S1}} + \frac{1}{\rho_{S2}}\right)}}$$

DOUBLE WALL result

At low frequencies $f < f_0$

Both walls move together (in phase) like one wall of twice the mass. So the mass law is recovered.

$$Z_w \simeq j\omega\left(\rho_{s1} + \rho_{s2}\right)$$

$$\mathrm{TL} \simeq 20\log\frac{\omega\left(\rho_{s1}\rho_{s2}\right)}{2\rho_0 c} \simeq 6\text{dB/octave}$$

At high frequencies $f > f_0$

Double walls are most effective.

$$Z_w \simeq -j\omega^3 \frac{\rho_{s1}\rho_{s2} d}{\rho_0 c^2}$$

$$\mathrm{TL} \simeq 20\log\frac{\omega^3\left(\rho_{s1}\rho_{s2}\right)}{2\rho_0^2 c^3} \simeq 18\text{dB/octave}$$

At very high frequencies $f << f_0$

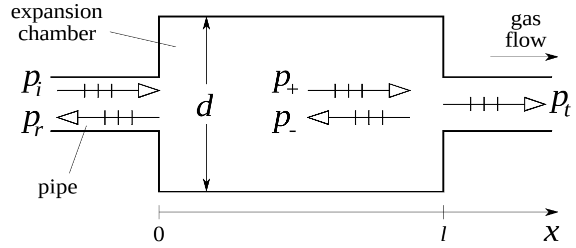
The walls decouple. The transmission loss is the sum of the losses of the two walls; there is no interaction.

$$\mathrm{TL} \simeq \mathrm{TL}_1 + \mathrm{TL}_2 \simeq 12\text{dB/octave}$$

MUFFLERS

EXPANSION CHAMBER

When sound traveling through a pipe encounters a section with a different cross-sectional area, it sees a new impedance and some sound is reflected. The dimensions can be chosen to optimize the transmission loss through the exit at particular frequencies. We assume $d < \lambda$.



Let $\quad p_i = P_i e^{j(\omega t - kx)} \qquad u_i = \dfrac{p_i}{\rho_0 c}$

$$p_r = P_r e^{j(\omega t + kx)} \qquad u_r = -\frac{p_r}{\rho_0 c}$$

$$p_+ = P_+ e^{j(\omega t - kx)} \qquad u_+ = \frac{p_+}{\rho_0 c}$$

$$p_- = P_- e^{j(\omega t + kx)} \qquad u_- = -\frac{p_-}{\rho_0 c}$$

$$p_t = P_t e^{j(\omega t - kx)} \qquad u_t = \frac{p_t}{\rho_0 c}$$

$$R = \frac{P_r}{P_i} \qquad T = \frac{P_t}{P_i} \qquad \alpha = \frac{P_+}{P_i} \qquad \beta = \frac{P_-}{P_i}$$

p_i = acoustic pressure, incident wave [Pa]
p_r = acoustic pressure, reflected wave [Pa]
p_+ = pressure, expan. chamber forward-traveling wave [Pa]
p_- = pressure, expan. chamber reverse-traveling wave [Pa]
p_t = acoustic pressure, transmitted wave [Pa]
P_i = peak acoustic pressure, incident [Pa]
P_r = peak acoustic pressure, reflected [Pa]
P_+ = peak acoustic pressure, expansion chamber forward-traveling wave [Pa]
P_- = peak acoustic pressure, expansion chamber reverse-traveling wave [Pa]
P_t = peak acoustic pressure, transmitted [Pa]
$\rho_0 c$ = impedance of the medium [rayls or (Pa·s)/m] (415 for air)
R = reflection coefficient [no units]
T = transmission coefficient [no units]
α = expansion exit transmission coefficient [no units]
β = expansion exit reflection coefficient [no units]
S_p = cross-sectional area of the pipe [m^2]
S_c = cross-sectional area of the expansion chamber [m^2]

Next, apply the boundary conditions:

EXPANSION CHAMBER BOUNDARY CONDITIONS

Boundary condition 1: at $x = 0$,

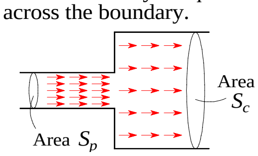
$$p_i + p_r = p_+ + p_- \;\rightarrow\; P_i + P_r = P_+ + P_- \;\rightarrow$$

$$\frac{P_i}{P_i} + \frac{P_r}{P_i} = \frac{P_+}{P_i} + \frac{P_-}{P_i} \;\rightarrow\; 1 + R = \alpha + \beta \quad \text{(i)}$$

Boundary condition 2: at $x = 0$,

Conservation of mass by equating volume velocities. The **volume velocity** is the cross-sectional area times the net velocity. *See* p8.

Volume velocity is equal across the boundary.



$$S_p(u_i + u_r) = S_c(u_+ + u_-) \rightarrow S_p(P_i - P_r) = S_c(P_+ - P_-)$$

$$\rightarrow \frac{P_i}{P_i} - \frac{P_r}{P_i} = \frac{S_c}{S_p}\left(\frac{P_+}{P_i} - \frac{P_-}{P_i}\right) \rightarrow 1 - R = m(\alpha - \beta) \quad \text{(ii)}$$

(i) + (ii): $\boxed{(1+m)\alpha + (1-m)\beta = 2}$ [Eqn. 1]

Boundary condition 3: at $x = l$,

$$p_+ + p_- = p_t \;\rightarrow\; P_+ e^{-jkl} + P_- e^{jkl} = P_t e^{-jkl} \;\rightarrow$$

$\boxed{\alpha e^{-jkl} + \beta e^{jkl} = T e^{-jkl}}$ [Eqn. 2]

Boundary condition 4: at $x = l$,

$$S_c(u_+ + u_-) = S_p u_t \;\rightarrow\; (P_+ + P_-) = \frac{S_p}{S_c} P_t$$

$\boxed{\alpha e^{-jkl} - \beta e^{jkl} = \tfrac{1}{m} T e^{-jkl}}$ [Eqn. 3]

m = the ratio of the cross-sectional area of the expansion chamber to the cross-sectional area of the pipe.

Next, solve the three equations:

EXPANSION CHAMBER, FINAL STEPS

Solve the three equations:

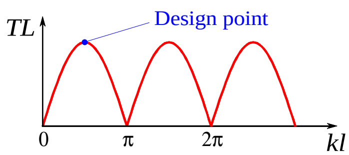
$$\begin{bmatrix} (1+m) & (1-m) & 0 \\ e^{-jkl} & e^{+jkl} & -e^{-jkl} \\ e^{-jkl} & -e^{+jkl} & -\frac{1}{m}e^{-jkl} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ T \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Cramer's Rule: $T = \frac{\Delta T}{\Delta} = \frac{e^{jkl}}{\cos kl + j\frac{1}{2}\left(m + \frac{1}{m}\right)\sin kl}$

$$TL = 10\log\frac{1}{T_I} = 10\log\frac{1}{|T|^2}$$

Transmission loss in an expansion chamber:

$$\boxed{TL = 10\log\left[1 + \tfrac{1}{4}\left(m - \tfrac{1}{m}\right)^2 \sin^2 kl\right]}$$



FLOW EFFECTS

Muffler performance is affected by flow rate, but the preceding calculations are valid for flows up to 35 m/s.

TEMPERATURE EFFECTS

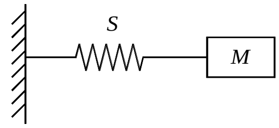
The effect of having high temperature gases in a muffler causes the speed of sound to increase, so λ becomes larger.

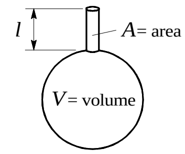
$$\boxed{\lambda = \frac{343}{f}\sqrt{\frac{T + 273}{293}}}$$

λ = wavelength [m]
f = frequency [Hz]
T = temperature [°C]

HELMHOLTZ RESONATOR (10.8)

A Helmholtz resonator is a vessel having a large volume with a relatively small neck. The gas in the neck looks like a lumped mass and the gas in the volume looks like a spring at low frequency.

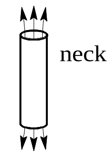




Stiffness due to a gas volume: $\boxed{s = \frac{\rho_0 c^2 A^2}{V}}$ [N/m]

Mass of the gas in neck: $m = \rho_0 l' A$ [kg]

Some gas spills out of the neck, so the mass plug is actually slightly longer than the neck. In practice, the effective length is:

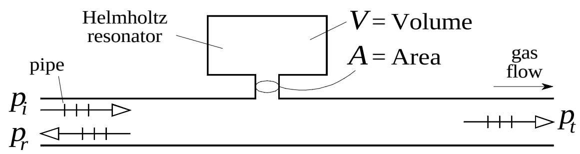


$$l' \approx l + 0.8\sqrt{A}$$

Resonance: $\omega_0 = \sqrt{\frac{S}{m}}, \quad f_0 = \frac{1}{2\pi}\sqrt{\frac{S}{m}} = \frac{c}{2\pi}\sqrt{\frac{A}{l'V}}$

SIDEBRANCH RESONATOR

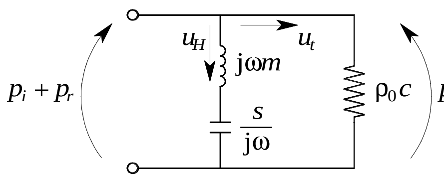
Refer to the Helmholtz resonator above.



The effect here is similar to the effect of blowing across the top of a coke bottle. The air across the bottle creates noise at many frequencies but the coke bottle responds only to its resonant frequency.

$$\boxed{\mathrm{TL} \approx 10\log\left[1+\left(\frac{\frac{\pi f_0 V}{sc}}{\frac{f}{f_0}-\frac{f_0}{f}}\right)^2\right]}$$

For a duct of impedance $\rho_0 c$ with a Helmholtz resonator having stiffness s and neck mass m, the arrangement can be modeled as follows.



- $\omega \to 0$, $\quad z \to \rho_0 c$
- $\omega \to \infty$, $\quad z \to \rho_0 c$
- $\omega \to \sqrt{s/m}$, $\quad z \to 0$

$$\boxed{z = \frac{\left(s-\omega^2 m\right)\rho_0 c}{s-\omega^2 m + j\omega\rho_0 c}}$$

TL = transmission loss [dB]
f = frequency [Hz]
f_0 = resonant frequency of Helmholtz resonator [Hz]
V = resonator volume [m^3]
s = stiffness [m^3]

ROOM ACOUSTICS

$\mathscr{E}$ ENERGY DENSITY [J/m^3] (5.8)

The amount of sound energy (potential and kinetic) per unit volume. In a perfectly diffuse field, $\mathscr{E}$ does not depend on location.

$$\boxed{\mathscr{E} = \frac{p_{\mathrm{rms}}^2}{\rho_0 c} = \frac{P^2}{2\rho_0 c}}$$

p_{rms} = acoustic pressure, rms [Pa]
P = peak acoustic pressure or pressure magnitude [Pa]
$\rho_0 c$ = impedance of the medium [rayls or (Pa·s)/m] (415 for air)

$\mathscr{E}(t)$ ROOM ENERGY DENSITY [J/m^3]

(12.2)

Sound growth: The following expression describes the effect of sound energy filling a room as a source is turned on at t=0.

$$\boxed{\mathscr{E}(t) = \frac{4W_0}{Ac}\left(1-e^{-t/\tau}\right)}$$

The following is the differential equation that describes the growth of sound energy in a live room.

$$\underbrace{V\frac{d\mathscr{E}}{dt}}_{\substack{\text{the rate at which}\\\text{energy increases}\\\text{in the volume}}} + \underbrace{\frac{Ac}{4}\mathscr{E}}_{\substack{\text{the rate at which}\\\text{energy is absorbed}\\\text{by the surfaces}}} = \underbrace{W_0}_{\substack{\text{power}\\\text{of the}\\\text{input}\\\text{source}}}$$

This can be rewritten to include the **time constant**.

$$\tau\frac{d\mathscr{E}}{dt}+\mathscr{E} = \frac{4W_0}{Ac}, \quad \text{where } \tau = \frac{4V}{Ac}$$

Sound decay: The following expression describes the effect of sound dissipation as a source is turned off at t=0.

$$\boxed{\mathscr{E}(t) = \mathscr{E}_0 e^{-t/\tau}}$$

W_0 = power of the sound source [W]
A = sound absorption, in units of *metric sabin* or *English sabin* [m^2 or ft^2]
t = time [s]
τ = time constant [s]
$\mathscr{E}_0$ = initial energy density [J/m^3]
c = the speed of sound (343 m/s in air) [m/s]

$\overline{\mathscr{E}}$ AVERAGE ENERGY DENSITY [J/m^3]

(12.2)

$$\boxed{\overline{\mathscr{E}} = \frac{1}{V}\int \mathscr{E}\, dV}$$

$\mathscr{E}$ = energy density [J/m^3]
V = room volume [m^3 or ft^3]

A ABSORPTION [m² or ft²] (12.1)

The absorption or *absorption area* may have units of metric sabins or English sabins, named for Wallace Sabine (1868-1919). The absorption area can be thought of as the equivalent area to be cut out of a wall in order to produce the same effect as an object of absorption A. See also *ABSORPTION COEFFICIENT* p21.

$$\boxed{A = \bar{a}S = \sum A_i}$$

$$\text{where} \quad \bar{a} = \frac{1}{S}\sum a_i S_i , \quad A_i = a_i S_i$$

$\bar{a}$ = average absorption coefficient [no units]
S = total surface area [m^2 or ft^2]
A_i = sound absorption of a particular material in the room [m^2 or ft^2]
a_i = absorption coefficient of a particular material [no units]
S_i = area represented by a particular material [no units]

$\bar{a}$ AVERAGE ABSORPTION [m² or ft²] (12.3)

The average sound absorption over an area.

$$\boxed{\bar{a} = \frac{1}{S}\sum a_i S_i}, \quad \bar{a} = \frac{W_{\text{abs}}}{W_{\text{incident}}} = \frac{A}{S}$$

A = total absorption area [m^2 or ft^2]
S = total surface area [m^2 or ft^2]
a_i = absorption coefficient of a particular material [no units]
S_i = area represented by a particular material [no units]
W_{abs} = power absorbed by the surfaces [W]
W_{incident} = power incident on the surfaces [W]

MEASURING ABSORPTION [m² or ft²]

The absorption of a sample can be measured by placing the sample in a reverberation chamber and measuring the effect it has on reverberation time. The absorption value for a person @ 1kHz is about 0.95 m^2, for a piece of furniture about 0.08 m^2. See also *ABSORPTION COEFFICIENT* p21.

$$\boxed{A_s = 0.161V\left(\frac{1}{T_s} - \frac{1}{T_0}\right)}$$

A_s = sound absorption of the sample [m^2 or ft^2]
V = volume of the room [m^3]
T_s = reverberation time with the sample in place [s]
T_0 = reverberation time in the empty room [s]

W_{abs} POWER ABSORBED [W] (12.2)

$$\boxed{W_{\text{abs}} = \bar{a}W_{\text{incident}}} \quad \text{where} \quad \bar{a} = \frac{1}{S}\sum a_i S_i$$

W_{incident} = power incident on the surface [W]
$\bar{a}$ = average absorption coefficient [no units]
S = total surface area [m^2 or ft^2]
A_i = sound absorption of a particular material in the room [m^2 or ft^2]
a_i = absorption coefficient of a particular material [no units]
S_i = area represented by a particular material [no units]

W_{incident} INCIDENT POWER [W] (12.2)

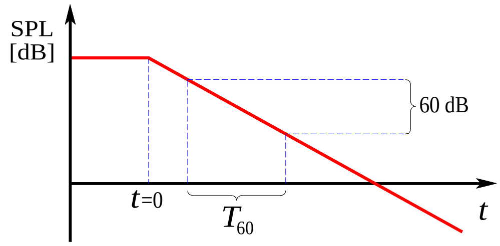
The total power incident on the walls of a room.

$$\boxed{W_{\text{incident}} = \frac{1}{4}Sc\bar{\mathscr{E}}}$$

S = total surface area [m^2 or ft^2]
c = the speed of sound (343 m/s in air) [m/s]
$\bar{\mathscr{E}}$ = average energy density [J/m^3]

T_{60} REVERBERATION TIME [s] (12.3)

The time required for a sound to decay by 60 dB, i.e. to one millionth of its previous value. Sound decay is linear when viewed on a log scale.



$$-60 = \frac{T}{\tau} 10 \log e \;\Rightarrow\; T = 13.816\tau, \quad \tau = \frac{4V}{ac}$$

Sabin formula:

$$T = \frac{0.161V}{A} \text{ (metric)}, \qquad T = \frac{0.049V}{A} \text{ (English)}$$

Including air absorption (for $\bar{a} \le 0.2$):

$$T = \frac{0.161V}{A + 4mV} \text{ (metric)}$$

More accurate, **Eyring-Norris reverberation formula**:

$$T = \frac{0.161V}{4mV - S\ln(1-\bar{a})} \text{ (metric)}$$

τ = time constant [s]
V = room volume [m^3 or ft^3]
A = sound absorption, in units of *metric sabin* or *English sabin* [m^2 or ft^2]
$\bar{a}$ = average absorption coefficient [no units]
m = air absorption coefficient [no units]
S = total surface area [m^2 or ft^2]

(p,q,r) MODES, rectangular cavity (9.1)

The modes of a volume are the frequencies at which resonances occur, and are a function of the room dimensions. For example, the lowest mode will be the frequency for which the longest dimension equals ½ -wavelength and is represented by (1,0,0).

$$f(p,q,r) = \frac{c}{2}\sqrt{\left(\frac{p}{L}\right)^2 + \left(\frac{q}{W}\right)^2 + \left(\frac{r}{H}\right)^2}$$

p, q, and r form the mode numbers. They are integers representing the number of half-wavelengths in the length, width, and height respectively. To avoid having more than one mode at the same frequency, the ratio of any two room dimensions should not be a whole number. Some recommended room dimension ratios are 1.6:1.25:1.0 for small rooms and 2.4:1.5:1.0 or 3.2:1.3:1.0 for large rooms.

f = frequency [Hz]
c = the speed of sound (343 m/s in air) [m/s]
L, W, H = room length, width, and height respectively [m]

$N(f)$ MODAL DENSITY [Hz^{-1}] (9.2)

The number of modes (resonant frequencies) per unit hertz. The modal density increases with frequency until it becomes a **diffuse field**. In a diffuse field, the modal structure is obscured and the sound field seems **isotropic**, i.e. the SPL is equal everywhere.

Rectangular room: $N(f) \approx \frac{4\pi}{c^3} V f^2$

f = frequency [Hz]
V = room volume [m^3]
c = the speed of sound (343 m/s in air) [m/s]

m AIR ABSORPTION COEFFICIENT, ARCHITECTURAL [no units] (12.3)

$$I = I_0 e^{-mx} = I_0 e^{-mct} \qquad m = 2\alpha$$

For most architectural applications, the air absorption coefficient can be approximated as:

$$m = 5.5 \times 10^{-4} (50/h)(f/1000)^{1.7}$$

I = acoustic intensity [W/m^2]
I_0 = initial acoustic intensity [W/m^2]
h = relative humidity (limited to the range 20 to 70%) [%]
f = frequency (limited to the range 1.5 to 10 kHz) [Hz]
c = the speed of sound (343 m/s in air) [m/s]
α = air absorption coefficient due to combined factors [no units]

L_M MEAN FREE PATH [m]

The average distance between reflections in a rectangular room. This works out to 2L/3 for a cubic room and 2d/3 for a sphere.

$$\boxed{L_M = \frac{4V}{S}}$$

V = room volume [m^3 or ft^3]
S = total surface area [m^2 or ft^2]

n NUMBER OF REFLECTIONS [no units]

The number of acoustic reflections in a room in time *t*.

$$n = \frac{ct}{L_M} = \frac{ctS}{4V}$$

c = the speed of sound (343 m/s in air) [m/s]
t = time [s]
L_M = mean free path [m]
V = room volume [m^3 or ft^3]
S = total surface area [m^2 or ft^2]

R ROOM CONSTANT [m^2]

Note that $R \simeq A$ when $\bar{a}$ is very small.

$$R = \frac{A}{1-\bar{a}} = \frac{\bar{a}S}{1-\bar{a}}$$

A = sound absorption, in units of *metric sabin* or *English sabin* [m^2 or ft^2]
$\bar{a}$ = average absorption coefficient [no units]
S = total surface area [m^2 or ft^2]

SPL SOUND POWER LEVEL [dB]

Source: $\text{SPL} = L_w = 10\log\frac{W}{W_{\text{ref}}}$

$$\text{SPL} = L_w + 10\log\left(\frac{Q}{4\pi r^2}+\frac{4}{R}\right) \underbrace{+10\log\left(\frac{\rho_0 c W_{\text{ref}}}{P_{\text{ref}}^2}\right)}_{\text{In mks units, this is 10 log 1.04, which is small enough to be ignored.}}$$

$$\boxed{\text{SPL} = L_w + 10\log\left(\frac{Q}{4\pi r^2}+\frac{4}{R}\right)}$$

L_w = sound power level of the source [dB]
W = sound power level of the source [W]
W_{ref} = reference power level, 10^{-12} [W]
Q = quality factor [no units]
r = distance from the source to the observation point [m]
R = room constant [m^2]
P_{ref} = the reference pressure 20×10^{-6} in air, 1×10^{-6} in water [Pa]

Q QUALITY FACTOR [no units]

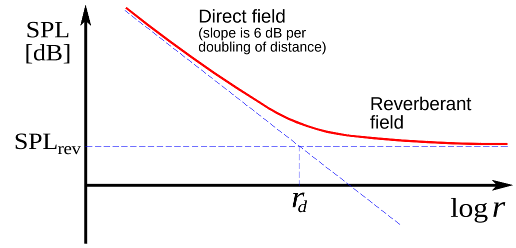
A factor that is dependent on the location of a source relative to reflective surfaces. The source strength or amplitude of volume velocity.

Q = 1 The source is located away from surfaces.
Q = 2 The source is located on a hard surface.
Q = 4 The source is located in a 2-way corner.

r = distance from the source to the observation point [m]
R = room constant [m^2]

DIRECT FIELD

The direct field is that part of a room in which the dominant sound comes directly (unreflected) from the source.



Energy density (direct)**:** $\mathscr{E}_{\text{dir}} = \frac{I}{c} = \frac{Q}{c}\frac{W}{4\pi r^2}$ [J/m^3]

I = acoustic intensity [W/m^2]
c = the speed of sound (343 m/s in air) [m/s]
Q = quality factor (Q=1 when source is remote from all surfaces) [no units]
W = sound power level of the source [W]
r = distance from the source to the observation point [m]

r_d REVERBERATION RADIUS [m]

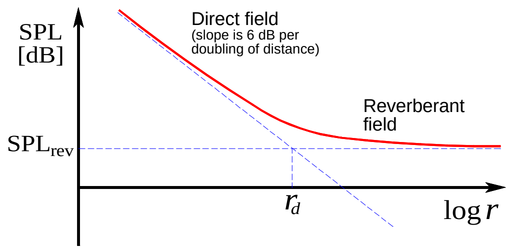
The distance from the source at which the SPL due to the source falls to the level of the reverberant field.

$$\boxed{r_d = \sqrt{\frac{QR}{16\pi}}}$$

Q = quality factor (Q=1 when source is remote from all surfaces) [no units]
R = room constant [m^2]

REVERBERANT FIELD

The area in a room that is remote enough from the sound source that movement within the field does not cause appreciable change in sound level. The area of the room not in the **direct field**.



Sound power level: $\mathrm{SPL}_{\mathrm{rev}} = L_w + 6 - 10\log R$ [dB]

Energy density (reverberant)**:** $\mathscr{E}_{\mathrm{rev}} = \dfrac{4W_{\mathrm{rev}}}{Ac} = \dfrac{4W}{Rc}$ [J/m^3]

$\mathrm{SPL}_{\mathrm{rev}}$ = sound power level in the reverberant field [m]
r = distance from the source to the observation point [m]
r_d = distance from the source to the reverberant field boundary [m]
L_w = sound power level of the source [dB]
R = room constant [m^2]
W_{rev} = reverberant sound power level of the room [W]
W = sound power level of the source [W]

NR NOISE REDUCTION [dB] (13.13)

The noise reduction from one room to an adjoining room is the difference between the sound power levels in the two rooms. The value is used in the measurement of transmission loss for various partition materials and construction.

$$\boxed{\mathrm{NR} = \mathrm{SPL}_1 - \mathrm{SPL}_2}$$

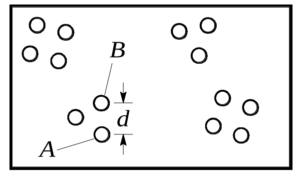
For measuring transmission loss:

$$\mathrm{TL} = \mathrm{NR} + 10\log\frac{S_w}{R_2}, \quad \textit{provided } R_2 \simeq A_2$$

S_w = surface area of the wall [m^2]
R_2 = room constant of the receiving room [m^2]
A_2 = sound absorption or *absorption area* of the receiving room [m^2]

COCKTAIL PARTY EFFECT

Consider a room of a given volume V and reverberation time T, and assume a fixed distance d among speakers and listeners in M small conversational groups. In each group, only one person is speaking at a time. There is a theoretical maximum number of groups that can exist before the onset of instability and loss of intelligibility. That is, as more conversations are added, one must speak louder in order to be heard. But with everyone speaking louder, the background noise increases, hence the instability.



Energy density at B due to speaker A:

$$\mathscr{E}_1 = \frac{W}{c}\left(\frac{1}{4\pi d^2} + \frac{4}{R}\right)$$

Reverberant energy density due to other *M*-1 conversations:

$$\mathscr{E}_{\mathrm{rev}} = (M-1)\frac{4W}{Rc}$$

Signal to noise ratio:

$$\mathrm{SNR} = \frac{\mathscr{E}_1}{\mathscr{E}_{\mathrm{rev}}} = \frac{1}{M-1}\left(\frac{R}{16\pi d^2} + 1\right)$$

Notice that the power W drops out of the equation.

Now if we require that this signal to noise ratio be some minimum required in order for the listener to be able to understand the speaker, the expression can be written:

$$M < 1 + \frac{1}{\mathrm{SNR}_{\min}}\left(\frac{R}{16\pi d^2} + 1\right)$$

Assume $R \simeq A$ so that $T = 0.161V/A \simeq 0.161V/R$, then we can rewrite the expression in terms of the room volume V and the reverberation constant T.

$$M < 1 + \frac{1}{\mathrm{SNR}_{\min}}\left(\frac{V}{312 d^2 T} + 1\right) \quad \text{[mks units]}$$

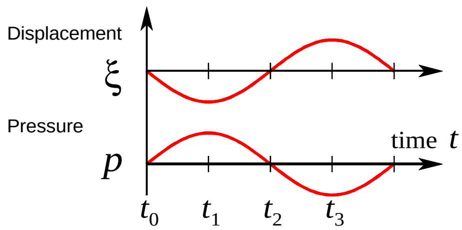
If we further assume that to the listener, the speaker must be as loud as the background noise, then the maximum number of speakers (conversations) in the room is

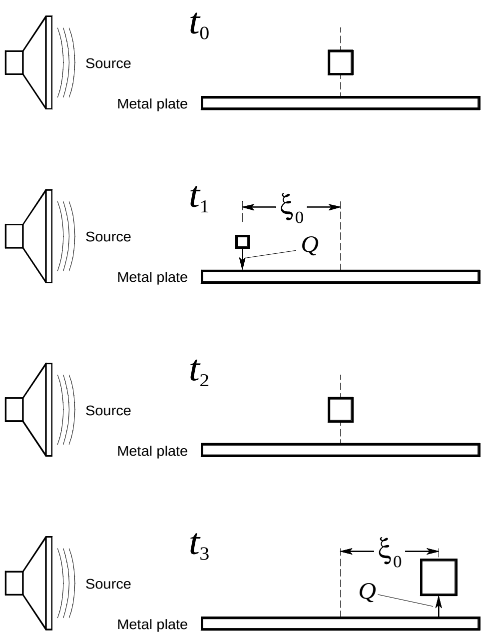
$$\boxed{M_{\max} \simeq 2 + \frac{V}{312 d^2 T}}$$

W = power output of a speaker [W]
M = the number of speakers (or groups)
R = room constant [m^2]
d = distance between speakers in the same group [m]
$\mathscr{E}$ = energy density [J/m^3]

THERMOACOUSTIC CYCLE

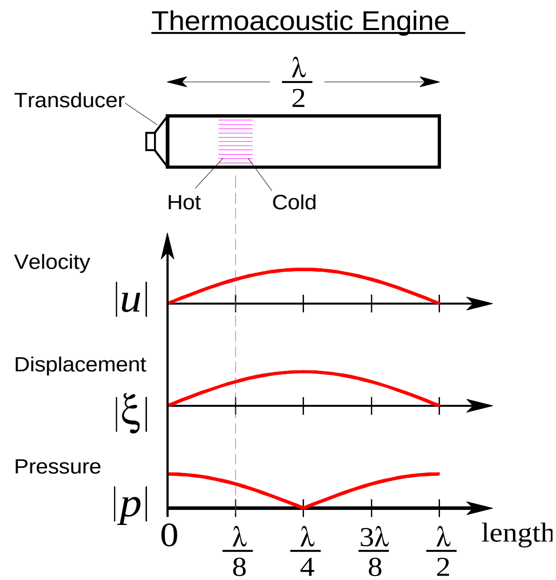
Consider a small volume of air in acoustic oscillation at t_0 and ambient pressure p_0. As it moves toward the source, pressure and temperature increase while volume decreases. The volume of air slows and reverses direction at t_1 and transfers heat to the metal plate. As the volume of air moves away from the source, pressure and temperature decrease. At the volume reaches t_3, it slows and again reverses direction. The cooler volume absorbs heat from the metal place. This action takes place all along the length of the metal plate, forming a *bucket brigade* of heat transfer.





THERMOACOUSTIC ENGINE

A transducer in one end of a half-wavelength chamber creates a high power standing wave. Thin metal plates are positioned ¼ of the way from one end so that velocity, displacement and pressure amplitudes will all be high.



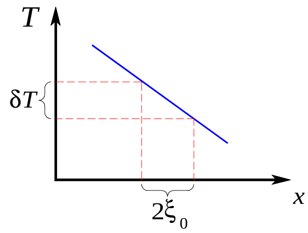
$$p = A\cos kx \sin \omega t\,, \quad k = \frac{\pi}{\lambda/2}\,, \quad \omega = \frac{\pi c}{\lambda/2}$$

$$u = -\frac{A}{\rho_0 c}\sin kx \cos \omega t$$

$$\xi = -\frac{A}{\omega \rho_0 c}\sin kx \cos \omega t$$

THERMOACOUSTIC GRADIENT

Also called **critical gradient**. The oscillatory motion and oscillatory temperature of gas particles along the metal plates establishes a temperature gradient along the plates. The parallel stacking of plates increases the power of the engine but does not affect the gradient.



Maximum temperature gradient:

$$\boxed{\left|\frac{dT}{dx}\right|_{\text{critical}} = (\gamma - 1) k T_0}$$

γ = ratio of specific heats (1.4 for a diatomic gas) [no units]
T_0 = ambient temperature [K]
k = wave number or propagation constant [rad./m]

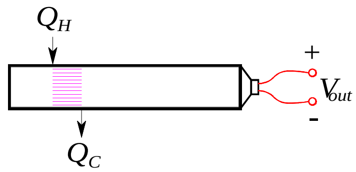
Γ GRADIENT RATIO

The ratio of the operating temperature gradient to the critical gradient.

$$\boxed{\Gamma = \frac{|dT/dx|}{|dT/dx|_{\text{critical}}}}$$

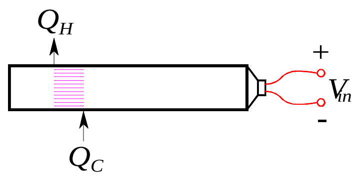
$\Gamma > 1$**: Thermoacoustic heat engine**

Heat flow generates sound (does work)



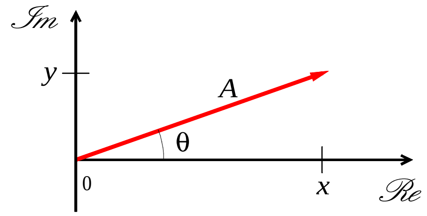
$\Gamma < 1$**: Thermoacoustic refrigerator**

Acoustic energy pumps heat from cold end to hot end of stack



GENERAL MATHEMATICAL

$x + \mathrm{j}\,y$ COMPLEX NUMBERS



$$x + \mathrm{j}y = Ae^{\mathrm{j}\theta} = A\cos\theta + \mathrm{j}A\sin\theta$$

$$\mathscr{Re}\{x + \mathrm{j}y\} = x = A\cos\theta$$

$$\mathscr{Im}\{x + \mathrm{j}y\} = y = A\sin\theta$$

$$\text{Magnitude}\{x + \mathrm{j}y\} = A = \sqrt{x^2 + y^2}$$

$$\text{Phase}\{x + \mathrm{j}y\} = \theta = \tan^{-1}\frac{y}{x}$$

$$j = e^{j\frac{\pi}{2}}$$

The magnitude of a complex number may be written as the absolute value.

$$\text{Magnitude}\{x + \mathrm{j}y\} = |x + \mathrm{j}y|$$

The square of the magnitude of a complex number is the product of the complex number and its **complex conjugate**. The **complex conjugate** is the expression formed by reversing the signs of the imaginary terms.

$$|x + \mathrm{j}y|^2 = (x + \mathrm{j}y)(x + \mathrm{j}y)^* = (x + \mathrm{j}y)(x - \mathrm{j}y)$$

PHASOR NOTATION

When the excitation is sinusoidal and under steady-state conditions, we can express a partial derivative in phasor notation, by replacing $\frac{\partial}{\partial t}$ with $j\omega$. For example, the Telegrapher's equation $\frac{\partial \mathscr{V}}{\partial z} = -L\frac{\partial \mathscr{I}}{\partial t}$ becomes $\frac{\partial V}{\partial z} = -Lj\omega I$. Note that $\mathscr{V}(z,t)$ and $\mathscr{I}(z,t)$ are functions of position and time (space-time functions) and $V(z)$ and $I(z)$ are functions of position only.

Sine and cosine functions are converted to exponentials in the phasor domain.

Example:

$$\vec{\mathscr{E}}(\vec{r},t) = 2\cos(\omega t + 3z)\,\hat{x} + 4\sin(\omega t + 3z)\,\hat{y}$$
$$= \mathscr{Re}\left\{2e^{\mathrm{j}3z}e^{\mathrm{j}\omega t}\,\hat{x} + (-\mathrm{j})\,4e^{\mathrm{j}3z}e^{\mathrm{j}\omega t}\,\hat{y}\right\}$$

$$\vec{E}(\vec{r}) = 2e^{\mathrm{j}3z}\,\hat{x} - \mathrm{j}4e^{\mathrm{j}3z}\,\hat{y}$$

TIME-AVERAGE

When two functions are multiplied, they cannot be converted to the phasor domain and multiplied. Instead, we convert each function to the phasor domain and multiply one by the complex conjugate of the other and divide the result by two. The **complex conjugate** is the expression formed by reversing the signs of the imaginary terms.

For example, the function for power is:

$$P(t) = v(t)\,i(t) \text{ watts}$$

Time-averaged power is:

$$\langle P(t)\rangle = \frac{1}{T}\int_0^T v(t)\,i(t)\,dt \text{ watts}$$

For a single frequency:

$$\langle P(t)\rangle = \frac{1}{2}\mathscr{Re}\{V\,I^*\} \text{ watts}$$

T = period [s]
V = voltage in the phasor domain [s]
I^* = complex conjugate of the phasor domain current [A]

RMS

rms stands for **root mean square**.

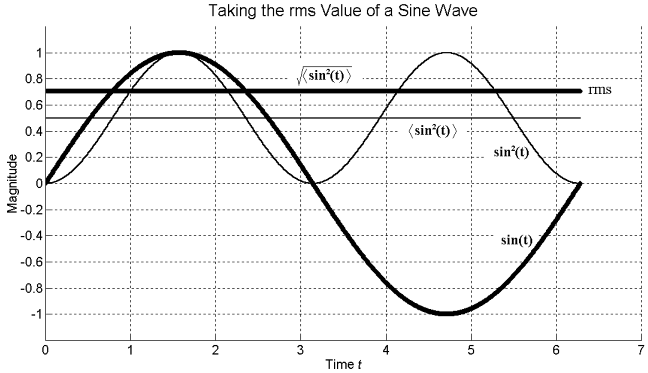
root mean square

$$\sqrt{\langle p^2 \rangle}$$

$$\boxed{f(t)_{\mathrm{rms}} = \sqrt{\left\langle f(t)^2 \right\rangle}}$$

The plot below shows a sine wave and its rms value, along with the intermediate steps of squaring the sine function and taking the mean value of the square. Notice that for this type of function, the mean value of the square is ½ the peak value of the square.

Taking the rms Value of a Sine Wave



EULER'S EQUATION

$$\boxed{e^{j\phi} = \cos\phi + j\sin\phi}$$

TRIGONOMETRIC IDENTITIES

$$e^{+j\theta} + e^{-j\theta} = 2\cos\theta$$

$$e^{+j\theta} - e^{-j\theta} = j2\sin\theta$$

$$e^{\pm j\theta} = \cos\theta \pm \mathrm{j}\sin\theta$$

CALCULUS

$$\int \sin^2 u\, du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$$

$$\int \cos^2 u\, du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$$

SERIES

$$\sqrt{1+x} \simeq 1 + \frac{1}{2}x, \quad |x| \ll 1$$

$$\frac{1}{\sqrt{1+x}} \simeq 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\frac{1}{1-x^2} \simeq 1 + x^2 + x^4 + x^6 + \cdots, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\frac{1}{(1-x)^2} \simeq 1 + 2x + 3x^2 + 4x^3 + \cdots, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\frac{1}{1+x} \simeq 1 - x + x^2 - x^3 + \cdots, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

$$\frac{1}{1-x} \simeq 1 + x + x^2 + x^3 + \cdots, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$$

BINOMIAL THEOREM

Also called binomial expansion. When m is a positive integer, this is a finite series of $m+1$ terms. When m is not a positive integer, the series converges for $-1<x<1$.

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + \cdots$$

BESSEL FUNCTION EXPANSION

$$\mathrm{J}_1: \quad \frac{z}{2} + \frac{2z^3}{2\cdot 4^2} - \frac{3z^5}{2\cdot 4^2 \cdot 6^2} + \cdots, \quad z \ll 1$$

HYPERBOLIC FUNCTIONS

$$j\sin\theta = \sinh(j\theta)$$

$$j\cos\theta = \cosh(j\theta)$$

$$j\tan\theta = \tanh(j\theta)$$

LINEARIZING AN EQUATION

Small nonlinear terms are removed. Nonlinear terms include:

- variables raised to a power
- variables multiplied by other variables

Δ values are considered variables, e.g. Δt.

DOT PRODUCT

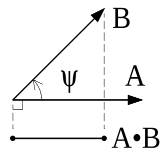
The dot product is a scalar value.

$$\mathbf{A}\bullet\mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z)\bullet(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\psi_{\mathrm{AB}}$$

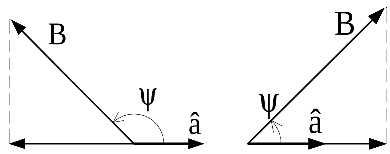
$$\hat{\mathbf{x}}\bullet\hat{\mathbf{y}} = 0,\quad \hat{\mathbf{x}}\bullet\hat{\mathbf{x}} = 1$$

$$\mathbf{B}\bullet\hat{\mathbf{y}} = (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)\bullet\hat{\mathbf{y}} = B_y$$



Projection of **B** along **â**:

$$(\mathbf{B}\bullet\hat{\mathbf{a}})\hat{\mathbf{a}}$$



The dot product of 90° vectors is zero.
The dot product is **commutative** and **distributive**:

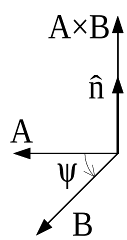
$$\mathbf{A}\bullet\mathbf{B} = \mathbf{B}\bullet\mathbf{A} \qquad \mathbf{A}\bullet(\mathbf{B}+\mathbf{C}) = \mathbf{A}\bullet\mathbf{B} + \mathbf{A}\bullet\mathbf{C}$$

CROSS PRODUCT

$$\mathbf{A}\times\mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z)\times(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)$$

$$= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x)$$

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}|\mathbf{A}||\mathbf{B}|\sin\psi_{\mathrm{AB}}$$



where $\hat{\mathbf{n}}$ is the unit vector normal to both **A** and **B** (thumb of right-hand rule).

$$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}$$

$$\mathbf{x}\times\mathbf{y} = \mathbf{z} \qquad \mathbf{y}\times\mathbf{x} = -\mathbf{z} \qquad \mathbf{x}\times\mathbf{x} = 0$$

$$\boldsymbol{\phi}\times\mathbf{z} = \mathbf{r} \qquad \boldsymbol{\phi}\times\mathbf{r} = -\mathbf{z}$$

The cross product is **distributive:**

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

Also, we have:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

Ñ NABLA, DEL OR GRAD OPERATOR

Compare the ∇ operation to taking the time derivative. Where $\partial/\partial t$ means to take the derivative with respect to time and introduces a s^{-1} component to the units of the result, the ∇ operation means to take the derivative with respect to distance (in 3 dimensions) and introduces a m^{-1} component to the units of the result. ∇ terms may be called space derivatives and an equation which contains the ∇ operator may be called a vector differential equation. In other words $\nabla\mathbf{A}$ is how fast **A** changes as you move through space.

in rectangular coordinates:
$$\nabla\mathbf{A} = \hat{x}\frac{\partial A}{\partial x} + \hat{y}\frac{\partial A}{\partial y} + \hat{z}\frac{\partial A}{\partial z}$$

in cylindrical coordinates:
$$\nabla\mathbf{A} = \hat{r}\frac{\partial A}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial A}{\partial \phi} + \hat{z}\frac{\partial A}{\partial z}$$

in spherical coordinates:
$$\nabla\mathbf{A} = \hat{r}\frac{\partial A}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial A}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial A}{\partial \phi}$$

Ñ² THE LAPLACIAN

The divergence of a gradient

Laplacian of a **scalar** in **rectangular** coordinates:
$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$

Laplacian of a **vector** in **rectangular** coordinates:
$$\nabla^2 \vec{A} = \hat{x}\frac{\partial^2 A_x}{\partial x^2} + \hat{y}\frac{\partial^2 A_y}{\partial y^2} + \hat{z}\frac{\partial^2 A_z}{\partial z^2}$$

In **spherical** and **cylindrical** coordinates:
$$\nabla^2\mathbf{A} \equiv \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$
$$= \mathrm{grad}(\mathrm{div}\ \mathbf{A}) - \mathrm{curl}(\mathrm{curl}\ \mathbf{A})$$

Ñ× DIVERGENCE

The del operator followed by the dot product operator is read as "the divergence of" and is an operation performed on a vector. In rectangular coordinates, ∇· means the sum of the partial derivatives of the magnitudes in the x, y, and z directions with respect to the x, y, and z variables. The result is a scalar, and a factor of m^{-1} is contributed to the units of the result.

For example, in this form of Gauss' law, where **D** is a density per unit area, $\nabla\cdot\mathbf{D}$ becomes a density per unit volume.

$$\boxed{\mathrm{div}\ \mathbf{D} = \nabla\cdot\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho}$$

D = electric flux density vector $\mathbf{D} = \varepsilon\mathbf{E}$ [$\mathrm{C/m^2}$]
ρ = source charge density [$\mathrm{C/m^3}$]

CURL $\text{curl}\,\mathbf{B} = \nabla \times \mathbf{B}$

The circulation around an enclosed area. The curl of vector $\mathbf{B}$ is

in **rectangular** coordinates:

$$\text{curl}\;\mathbf{B} = \nabla \times \mathbf{B} =$$

$$\hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

in **cylindrical** coordinates:

$$\text{curl}\;\mathbf{B} = \nabla \times \mathbf{B} =$$

$$\hat{r}\left[\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right] + \hat{z}\frac{1}{r}\left[\frac{\partial\left(rB_\phi\right)}{\partial r} - \frac{\partial B_r}{\partial \phi}\right]$$

in **spherical** coordinates:

$$\text{curl}\;\mathbf{B} = \nabla \times \mathbf{B} = \hat{r}\;\frac{1}{r\sin\theta}\left[\frac{\partial\left(B_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial B_\theta}{\partial \phi}\right] +$$

$$\hat{\theta}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{\partial\left(rB_\phi\right)}{\partial r}\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial\left(rB_\theta\right)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right]$$

The **divergence of a curl** is always zero:

$$\nabla \cdot \left(\nabla \times \mathbf{H}\right) = 0$$

SPHERE

$\text{Area} = \pi d^2 = 4\pi r^2$ $\qquad$ $\text{Volume} = \tfrac{1}{6}\pi d^3 = \tfrac{4}{3}\pi r^3$

GRAPHING TERMINOLOGY

With *x* being the horizontal axis and *y* the vertical, we have a graph of ***y* versus *x*** or ***y* as a function of *x***. The *x*-axis represents the **independent variable** and the *y*-axis represents the **dependent variable**, so that when a graph is used to illustrate data, the data of regular interval (often this is time) is plotted on the *x*-axis and the corresponding data is dependent on those values and is plotted on the *y*-axis.

GLOSSARY

adiabatic Occurring without loss or gain or heat.

anechoic room Highly absorptive room. $a \approx 1$.

enthalpy (H) A thermodynamic property. The sum of the internal energy U and the volume-pressure product PV. If a body is heated without changing its volume or pressure, then the change in enthalpy will equal the heat transfer. Units of kJ. Enthalpy also refers to the more commonly used specific enthalpy or enthalpy per unit mass h, which has units of kJ/kg.

entropy A measure of the unavailable energy in a closed thermodynamic system, varies in direct proportion to temperature change of the system. The *thermal charge*.

harmonic wave A waveform that is sinusoidal in time.

isentropic Having constant entropy, no change in thermal charge. However there could be heat flow in and out, analogous to current flow.

isothermal Having constant temperature, no heat flow to/from the surroundings. Analogous to voltage.

pink noise Noise composed of all audible frequencies with a 3 dB per octave attenuation with frequency increase. The attenuation is based on a per Hz value; the SPLs for each *octave* are equal.

reverberation room Characterized by long decay time. $a_0 << 1$, large T_0..

TDS time delay spectrometry. A sophisticated method for obtaining anechoic results in echoic spaces.

white noise Noise composed of all audible frequencies at equal amplitude per Hz.

For a more comprehensive glossary, see the file DictionaryOfAcousticTerms.PDF.