ANTENNAS AND WIRELESS PROPAGATION EE325K

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SOME BASICS

COULOMB

A unit of electrical charge equal to one amp second, the charge on 6.21×10^{18} electrons, or one joule per volt.

UNITS, electromagnetics

$$\left.\begin{array}{ll} \vec{\mathscr{E}} & \text{electric field [V/m]} \\ \vec{\mathscr{H}} & \text{magnetic field [A/m]} \\ \vec{\mathscr{D}} & \text{electric flux density [C/m}^2\text{]} \\ \vec{\mathscr{B}} & \text{magnetic flux density [Wb/m}^2\text{]} \end{array}\right\}\text{fields}$$

$$\left.\begin{array}{ll} \vec{\mathscr{J}} & \text{electric current density [A/m}^2\text{]} \\ \rho_{\varepsilon} & \text{electric charge density [C/m}^3\text{]} \end{array}\right\}\text{sources}$$

These vectors are a function of space and time $(\vec{r}, t)$.

UNITS, electrical

I (current in amps) = $\frac{q}{s}=\frac{W}{V}=\frac{J}{V\cdot s}=\frac{N\cdot m}{V\cdot s}=\frac{V\cdot C}{s}$

q (charge in coulombs) = $I\cdot s=V\cdot C=\frac{J}{V}=\frac{N\cdot m}{V}=\frac{W\cdot s}{V}$

C (capacitance in farads) = $\frac{q}{V}=\frac{q^2}{J}=\frac{q^2}{N\cdot m}=\frac{J}{V^2}=\frac{I\cdot s}{V}$

H (inductance in henrys) = $\frac{V\cdot s}{I}$ (note that $\mathrm{H}\cdot\mathrm{F}=s^2$)

J (energy in joules) =

$$N\cdot m=V\cdot q=W\cdot s=I\cdot V\cdot s=C\cdot V^2=\frac{q^2}{C}$$

N (force in newtons) = $\frac{J}{m}=\frac{q\cdot V}{m}=\frac{W\cdot s}{m}=\frac{kg\cdot m}{s^2}$

T (magnetic flux density in teslas) = $\frac{Wb}{m^2}=\frac{V\cdot s}{m^2}=\frac{H\cdot I}{m^2}$

V (electric potential in volts) =

$$\frac{W}{I}=\frac{J}{q}=\frac{J}{I\cdot s}=\frac{W\cdot s}{q}=\frac{N\cdot m}{q}=\frac{q}{C}$$

W (power in watts) =

$$\frac{J}{s}=\frac{N\cdot m}{s}=\frac{q\cdot V}{s}=V\cdot I=\frac{C\cdot V^2}{s}=\frac{1}{746}HP$$

Wb (magnetic flux in webers) = $H\cdot I=V\cdot s=\frac{J}{I}$

where s is seconds

l WAVELENGTH [m]

The distance that a wave travels during one cycle.

$$\boxed{\lambda=\frac{v_p}{f}=\frac{2\pi}{k}}$$

v_p = velocity of propagation (speed of light 2.998×10^8 m/s in free space)

f = frequency [Hz]

$k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}$, the wave number or propagation constant [m^{-1}]

For complex k, ($k=k'-jk''$):

$$\lambda=\frac{2\pi}{k'}$$

The relation at right can be used to quickly approximate radio frequency wavelengths. For example, at 300 MHz it is easily seen that the wavelength is 1m.

$$\lambda=\frac{300}{f\,(\mathrm{MHz})}$$

CONSTANTS

Avogadro's number [molecules/mole]	$N_A=6.02\times10^{23}$
Boltzmann's constant	$k=1.38\times10^{-23}$ J/K
	$=8.62\times10^{-5}$ eV/K
Elementary charge	$q=1.60\times10^{-19}$ C
Electron mass	$m_0=9.11\times10^{-31}$ kg
Permittivity of free space	$\varepsilon_0=8.85\times10^{-12}$ F/m
Permeability constant	$\mu_0=4\pi\times10^{-7}$ H/m
Planck's constant	$h=6.63\times10^{-34}$ J-s
	$=4.14\times10^{-15}$ cV-s
Rydberg constant	$R=109{,}678$ cm^{-1}
kT @ room temperature	$kT=0.0259$ eV
Speed of light	$c=2.998\times10^8$ m/s
1 Å (angstrom)	10^{-8} cm = 10^{-10} m
1 µm (micron)	10^{-4} cm
1 nm = 10Å = 10^{-7} cm	
1 eV = 1.6 × 10^{-19} J	

ELECTROMAGNETIC SPECTRUM

FREQUENCY	WAVELENGTH (free space)	DESIGNATION	APPLICATIONS
< 3 Hz	> 100 Mm		Geophysical prospecting
3-30 Hz	10-100 Mm	ELF	Detection of buried metals
30-300 Hz	1-10 Mm	SLF	Power transmission, submarine communications
0.3-3 kHz	0.1-1 Mm	ULF	Telephone, audio
3-30 kHz	10-100 km	VLF	Navigation, positioning, naval communications
30-300 kHz	1-10 km	LF	Navigation, radio beacons
0.3-3 MHz	0.1-1 km	MF	AM broadcasting
3-30 MHz	10-100 m	HF	Short wave, citizens' band
30-300 MHz 54-72 76-88 88-108 174-216	1-10 m	VHF	TV, FM, police TV channels 2-4 TV channels 5-6 FM radio TV channels 7-13
0.3-3 GHz 470-890 MHz 915 MHz 800-2500 MHz 1-2 2.45 2-4	10-100 cm	UHF "money band"	Radar, TV, GPS, cellular phone TV channels 14-83 Microwave ovens (Europe) PCS cellular phones, analog at 900 MHz, GSM/CDMA at 1900 L-band, GPS system Microwave ovens (U.S.) S-band
3-30 GHz 4-8 8-12 12-18 18-27	1-10 cm	SHF	Radar, satellite communications C-band X-band (Police radar at 11 GHz) K_u-band (dBS Primestar at 14 GHz) K-band (Police radar at 22 GHz)
30-300 GHz 27-40 40-60 60-80 80-100	0.1-1 cm	EHF	Radar, remote sensing K_a-band (Police radar at 35 GHz) U-band V-band W-band
0.3-1 THz	0.3-1 mm	Millimeter	Astromony, meteorology
10^{12}-10^{14} Hz	3-300 μm	Infrared	Heating, night vision, optical communications
3.95×10^{14}- 7.7×10^{14} Hz	390-760 nm 625-760 600-625 577-600 492-577 455-492 390-455	Visible light	Vision, astronomy, optical communications Red Orange Yellow Green Blue Violet
10^{15}-10^{18} Hz	0.3-300 nm	Ultraviolet	Sterilization
10^{16}-10^{21} Hz		X-rays	Medical diagnosis
10^{18}-10^{22} Hz		γ-rays	Cancer therapy, astrophysics
> 10^{22} Hz		Cosmic rays	Astrophysics

ELECTROMAGNETIC WAVES

THE ELECTROMAGNETIC SPECTRUM

The HF band is useful for longrange communications because these frequencies tend to bounce off the ionosphere (a atmospheric layer whose lower boundary is about 30 miles up) and the earth's surface. Only three transmitters would be required for global coverage. The effect varies with time of day due to the effects of sunlight on the ionosphere. The ionosphere is transparent to the FM band.

The attenuation effect of the atmosphere peaks at various frequencies, notably at 60 GHz due to oxygen. This frequency is used for intersatellite communications when it is desired that the signal not reach earth.

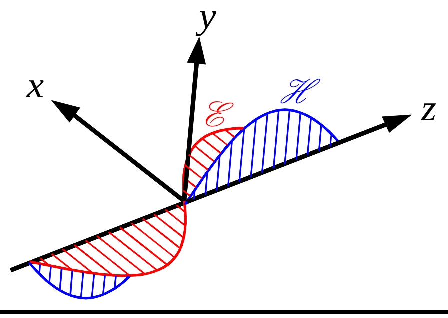
The attenuation effect of the atmosphere is at a low point at 94 GHz, which is in the W-band, used for radar.

The frequency of U.S. microwave ovens (2.45 GHz) is an ISM frequency (Industrial, Scientific, Medical).

At the high frequencies, the electromagnetic spectrum becomes more particle-like and less wave-like.

VISUALIZING THE ELECTROMAGNETIC WAVE

The electric field $\mathscr{E}$ and the magnetic field $\mathscr{H}$ are at right angles to each other. With the electric field aligned with the x-axis and the magnetic field on the y-axis, propagation is in the z-direction. Propagation is the movement of the *effect* of the electromagnetic disturbance. This is analogous to dropping a pebble in a pond. The ripples propagate outward from the source of the disturbance but the water only moves in vertical oscillation.



h CHARACTERISTIC IMPEDANCE [Ω]

The characteristic or intrinsic wave impedance is the ratio of electric to magnetic field components, a characteristic of the medium. η_0 is the **characteristic impedance of free space** with a value of 377Ω.

$$\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}}$$

μ = permeability [H/m]
ε = permittivity [F/m]

The characteristic impedance can be used to relate the electric and magnetic fields.

$$\boxed{\vec{E} = -\eta\left(\hat{k}\times\vec{H}\right)} \qquad \boxed{\vec{H} = \frac{1}{\eta}\left(\hat{k}\times\vec{E}\right)}$$

Other relations:

$$\omega\mu = k\eta \qquad \eta\omega\varepsilon = k$$

$\hat{k}$ = a unit vector in the direction of propagation

k PHASE CONSTANT [m^{-1}]

The **wave number** or **propagation constant** or **phase constant** k (sometimes called β) depends on the source frequency and characteristics of the medium. $k=\omega\sqrt{\mu\varepsilon}$ is called the dispersion relationship.

$$\boxed{k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}=\frac{\omega}{c}=\omega\varepsilon\eta}$$

To obtain the **complex phase constant**, use the complex permittivity ε_{new}, see page 12. $k=\omega\sqrt{\mu\varepsilon_{\text{new}}}=k'-\mathrm{j}k''$ The real part k' (always positive) governs the propagation and the imaginary part k'' (always subtracted) governs the damping. The exponential term of the wave equation looks like this:

+z propagating: $e^{-\mathrm{j}kz}=e^{-\mathrm{j}(k'-\mathrm{j}k'')z}=e^{-\mathrm{j}k'z}e^{-k''z}$

-z propagating: $e^{+\mathrm{j}kz}=e^{+\mathrm{j}(k'-\mathrm{j}k'')z}=e^{+\mathrm{j}k'z}e^{+k''z}$

Note that the exponent containing k'' will always have the same sign as the exponent containing k', since wave decay must occur in the same direction as wave propagation. It is important to remember that when reading the complex phase constant from a wave expression that the k' and k'' terms are themselves always positive and the + or – signs in the exponents are associated with z, determining the direction of propagation.

Complex phase constant in the time domain:

$$\mathscr{Re}\left\{e^{-\mathrm{j}k'z}e^{-k''z}e^{\mathrm{j}\omega t}\right\}=\underbrace{\cos\left(\omega t-k'z\right)}_{+z\text{ propagation}}\underbrace{e^{-k''z}}_{\text{damps in }+z\text{ direction}}$$

ω = angular frequency of the source [radians/s]
ε_0 = permittivity of free space 8.85 × 10^{-12} [F/m]
μ_0 = permeability of free space $4\pi\times10^{-7}$ [H/m]
$\lambda=\frac{2\pi}{k}=\frac{c}{f}$, the wavelength [m]

$\vec{k}$ WAVE VECTOR [m^{-1}]

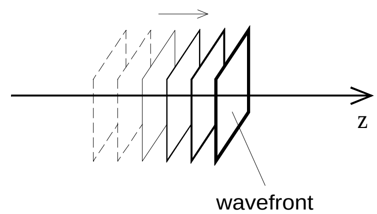
The phase constant k is converted to a vector. The vector $\vec{k}$ is in the direction of propagation.

$$\vec{k}=k\hat{k}=k_x\hat{x}+k_y\hat{y}+k_z\hat{z}$$

$k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}$, the wave number or propagation constant [rad./m]

PLANE WAVE

A pebble dropped in a pond produces a circular wave. A plane wave presents a planar wavefront. Function variables within a plane have uniform amplitude and phase values. Plane waves do not exist in nature but the idea is useful as an approximation in some circumstances. A radio wave at great distance from the transmitting antenna could be considered a plane wave. So treating a wave as a plane wave is to ignore the source, hence they are also called "source-free" waves.



Linearly polarized electromagnetic wave equations:

Electric field: $\vec{E}_x=\hat{x}E_0\cos\left(\omega t-kz\right)$

$\vec{E}_x=\hat{x}E_0e^{-\mathrm{j}kz}$ (phasor form)

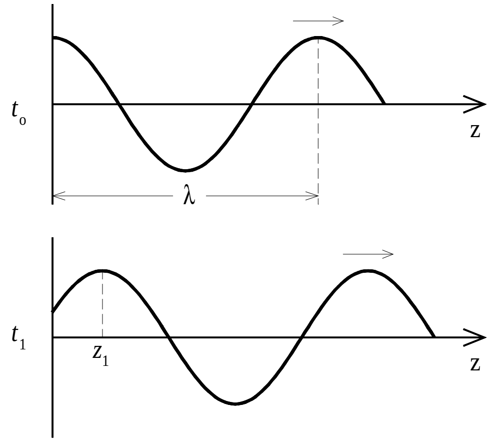
Magnetic field: $\vec{H}_y=\hat{y}\frac{E_0}{\eta}\cos\left(\omega t-kz\right)$

$\vec{H}_y=\hat{y}\frac{E_0}{\eta}e^{-\mathrm{j}kz}$ (phasor form)

ω = angular frequency of the source [radians/s]
E_0 = peak amplitude [V/m]
t = time [s]
$k=\omega\sqrt{\mu\varepsilon}=\frac{2\pi}{\lambda}$, the wave number or propagation constant [rad./m]
z = distance along the axis of propagation [m]
$\eta=\sqrt{\mu/\varepsilon}$ intrinsic wave impedance, the ratio of electric to magnetic field components, a characteristic of the medium [Ω]

TRAVELING/PROPAGATING WAVE

A wave traveling in the $+z$ direction can be expressed e^{-jkz} and a wave in the opposite direction would be expressed e^{+jkz}.



$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$, the wave number or propagation constant [m^{-1}]

$\lambda = \frac{2\pi}{k} = \frac{c}{f}$, the wavelength [m]

THE COMPLEX WAVE EQUATION

The **complex wave equation** is applicable when the excitation is sinusoidal and under steady state conditions.

$$\boxed{\frac{d^2V(z)}{dz^2} + k^2V(z) = 0}$$

where $k = \omega\sqrt{\mu_0\varepsilon_0} = \frac{2\pi}{\lambda}$ is the **phase constant** and is often represented by the letter β.

The *complex wave equation* above is a **second-order ordinary differential equation** commonly found in the analysis of physical systems. The general solution is:

$$\boxed{V(z) = V^+e^{-jkz} + V^-e^{+jkz}}$$

where e^{-jkz} and e^{+jkz} represent wave propagation in the $+z$ and $-z$ directions respectively.

PHASOR NOTATION

When the excitation is sinusoidal and under steady-state conditions, we can convert between the time domain and the phasor domain.

Where $z(t)$ is a function in the time domain and Z is its equivalent in the phasor domain, we have $z(t) = \mathscr{Re}\{Ze^{j\omega t}\}$.

Time Domain $\boldsymbol{Z(t)}$		**Phasor Domain** $\boldsymbol{Z}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t + \phi_0)$	$\leftrightarrow$	$Ae^{j\phi_0}$
$Ae^{-\alpha x}\cos(\omega t + \phi_0)$	$\leftrightarrow$	$Ae^{-\alpha x}e^{j\phi_0}$
$A\sin\omega t$	$\leftrightarrow$	$-jA$
$A\sin(\omega t + \phi_0)$	$\leftrightarrow$	$-jAe^{j\phi_0}$
$\frac{d}{dt}[z(t)]$	$\leftrightarrow$	$j\omega Z$
$\int z(t)\,dt$	$\leftrightarrow$	$\frac{1}{j\omega}Z$

Example, time domain to phasor domain:

$$\vec{\mathscr{E}}(\vec{r}, t) = 2\cos(\omega t + kz)\hat{x} + 4\sin(\omega t + kz)\hat{y}$$
$$= \mathscr{Re}\left\{2e^{jkz}e^{j\omega t}\hat{x} + (-j)4e^{jkz}e^{j\omega t}\hat{y}\right\}$$

$$\vec{E}(\vec{r}) = 2e^{jkz}\hat{x} - j4e^{jkz}\hat{y}$$

TIME-AVERAGE

When two functions are multiplied, they cannot be converted to the phasor domain and multiplied. Instead, we convert each function to the phasor domain and multiply one by the complex conjugate of the other and divide the result by two.

For example, the function for power is:

$P(t) = v(t)\,i(t)$ watts

Time-averaged power is:

$\langle P(t)\rangle = \frac{1}{T}\int_0^T v(t)\,i(t)\,dt = \frac{1}{2}\mathscr{Re}\{V I^*\}$ watts

T = period [s]
V = voltage in the phasor domain [V]
I^* = complex conjugate of the phasor domain current [A]

STANDING WAVE RATIO

$$\boxed{\mathrm{SWR} = \frac{|V|_{\max}}{|V|_{\min}} = \frac{|I|_{\max}}{|I|_{\min}} = \frac{1+\rho}{1-\rho}}$$

MAXWELL'S EQUATIONS

Maxwell's equations govern the principles of guiding and propagation of electromagnetic energy and provide the foundations of all electromagnetic phenomena and their applications. The time-harmonic expressions can be used only when the wave is sinusoidal.

	STANDARD FORM (Time Domain)	**TIME-HARMONIC** (Frequency Domain)
Faraday's Law	$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t}$	$\nabla \times \vec{E} = -j\omega \vec{B}$
Ampere's Law*	$\nabla \times \vec{\mathscr{H}} = \vec{\mathscr{J}} + \frac{\partial \vec{\mathscr{D}}}{\partial t}$	$\nabla \times \vec{H} = j\omega \vec{D} + \vec{J}$
Gauss' Law	$\nabla \cdot \vec{\mathscr{D}} = \rho_v$	$\nabla \cdot \vec{D} = \rho_v$
no name law	$\nabla \cdot \vec{\mathscr{B}} = 0$	$\nabla \cdot \vec{B} = 0$

$\mathscr{E}$ = electric field [V/m]
$\mathscr{B}$ = magnetic flux density [Wb/m^2 or T] $\mathscr{B} = \mu_0\mathscr{H}$
t = time [s]
$\mathscr{D}$ = electric flux density [C/m^2] $\mathscr{D} = \varepsilon_0\mathscr{E}$
ρ = volume charge density [C/m^3]
$\mathscr{H}$ = magnetic field intensity [A/m]
$\mathscr{J}$ = current density [A/m^2]

*Maxwell added the $\frac{\partial \vec{\mathscr{D}}}{\partial t}$ term to Ampere's Law.

MAXWELL'S EQUATIONS
source-free or plane wave solution

If we consider an electromagnetic wave at some distance from the source, it can be approximated as a plane wave, that is, having a planar wavefront rather than spherical shape. In this approximation, the source components of Maxwell's equations can be ignored and the equations become:

SOURCE-FREE (Time-dependent)	**SOURCE-FREE** (Time-harmonic)
$\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t}$	$\nabla \times \vec{E} = -j\omega \vec{B}$
$\nabla \times \vec{\mathscr{H}} = \frac{\partial \vec{\mathscr{D}}}{\partial t}$	$\nabla \times \vec{H} = j\omega \vec{D} = j\omega\varepsilon \vec{E}$
$\nabla \cdot \vec{\mathscr{D}} = 0$	$\nabla \cdot \vec{D} = 0$
$\nabla \cdot \vec{\mathscr{B}} = 0$	$\nabla \cdot \vec{B} = 0$

FARADAY'S LAW

When the magnetic flux enclosed by a loop of wire changes with time, a current is produced in the loop. The variation of the magnetic flux can result from a time-varying magnetic field, a coil in motion, or both..

$$\boxed{\nabla \times \vec{\mathscr{E}} = -\frac{\partial \vec{\mathscr{B}}}{\partial t}}$$

$\nabla \times \vec{\mathscr{E}}$ = the curl of the electric field
$\mathscr{B} = \mu_0\mathscr{H}$ magnetic flux density [Wb/m^2 or T]

Another way of expressing Faraday's law is that a changing magnetic field induces an electric field.

$$V_{ind} = \oint_C \vec{\mathscr{E}} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{\mathscr{B}} \cdot d\vec{s}$$

where S is the surface enclosed by contour C.

GAUSS'S LAW

The net flux passing through a surface enclosing a charge is equal to the charge. Careful, what this first integral really means is the surface area multiplied by the perpendicular electric field. There may not be any integration involved.

$$\boxed{\oint_S \varepsilon_0 \vec{\mathscr{E}} \cdot d\mathbf{s} = Q_{enc}} \qquad \oint_S \vec{\mathscr{E}} \cdot d\mathbf{s} = \int_V \rho \; dv = Q_{enc}$$

ε_0 = permittivity of free space 8.85 × 10^{-12} F/m
$\mathscr{E}$ = electric field [V/m]
$\mathscr{D}$ = electric flux density [C/m^2]
$d\mathbf{s}$ = a small increment of surface S
ρ = volume charge density [C/m^3]
dv = a small increment of volume V
Q_{enc} = total electric charge enclosed by the Gaussian surface [S]

The differential version of Gauss's law is:

$$\boxed{\nabla \cdot \vec{\mathscr{D}} = \rho} \quad \text{or} \quad \boxed{\operatorname{div}\left(\varepsilon \vec{\mathscr{E}}\right) = \rho}$$

GAUSS'S LAW – an example problem

Find the intensity of the electric field at distance r from a straight conductor having a voltage V.

Consider a cylindrical surface of length l and radius r enclosing a portion of the conductor. The electric field passes through the curved surface of the cylinder but not the ends. Gauss's law says that the electric flux passing through this curved surface is equal to the charge enclosed.

$$\oint_S \varepsilon_0 \mathscr{E} \cdot d\mathbf{s} = \varepsilon_0 \int_0^{2\pi} E_r l r \;\; d\phi = Q_{enc} = \rho_l l = C_l V l$$

so $\varepsilon_0 E_r r \int_0^{2\pi} d\phi = C_l V$ and $E_r = \dfrac{C_l V}{2\pi\varepsilon_0 r}$

E_r = electric field at distance r from the conductor [V/m]
l = length [m]
$r\,d\phi$ = a small increment of the cylindrical surface S [m^2]
ρ_l = charge density per unit length [C/m]
C_l = capacitance per unit length [F/m]
V = voltage on the line [V]

CONSTITUTIVE RELATIONS

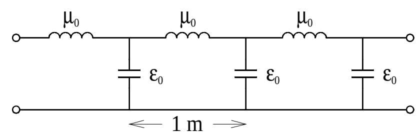
$$\left(\vec{\mathscr{D}}, \vec{\mathscr{B}}\right) \Leftrightarrow \left(\vec{\mathscr{E}}, \vec{\mathscr{H}}\right)$$

The *magnetic field intensity vector* $\vec{\mathscr{H}}$ is directly analogous to the *electric flux density vector* $\vec{\mathscr{D}}$ in electrostatics in that both $\vec{\mathscr{D}}$ and $\vec{\mathscr{H}}$ are medium-independent and are directly related to their sources.

In free space: ... $\vec{\mathscr{D}} = \varepsilon_0 \vec{\mathscr{E}} \qquad \vec{\mathscr{B}} = \mu_0 \vec{\mathscr{H}}$

$\mathscr{D}$ = electric flux density [C/m^2]
$\mathscr{E}$ = electric field [V/m]
$\mathscr{B}$ = magnetic flux density [Wb/m^2 or T]
$\mathscr{H}$ = magnetic field intensity [A/m]
ε_0 = permittivity of free space 8.85 × 10^{-12} [F/m]
μ_0 = permeability of free space $4\pi \times 10^{-7}$ [H/m]

Free space looks like a transmission line:



CONSERVATIVE FIELD LAW

$$\nabla \times \vec{\mathscr{E}} = 0 \qquad \oint_S \mathscr{E} \cdot d\mathbf{l} = 0$$

$\mathscr{E}$ = vector electric field [V/m]
$d\mathbf{l}$ = a small increment of length

EQUATION OF CONTINUITY

The differential form of the law of **conservation of charge**.

$$\boxed{\nabla \cdot \vec{\mathscr{J}} = -\frac{\partial \rho}{\partial t}}$$

In expanded phaser form: $\dfrac{\partial J_x}{\partial x} + \dfrac{\partial J_y}{\partial y} + \dfrac{\partial J_z}{\partial z} = -j\omega\rho$

$\mathscr{J}$ = current density [A/m^2]
ρ = volume charge density [C/m^3]
t = time [s]

COULOMB'S LAW

$$\nabla \cdot \vec{\mathscr{D}} = \rho \qquad \oint_S \vec{\mathscr{D}} \cdot d\mathbf{s} = \int_V \rho \;\; dv$$

$\mathscr{D}$ = electric flux density [C/m^2]
ρ = volume charge density [C/m^3]
$d\mathbf{s}$ = a small increment of surface S

POYNTING'S THEOREM

Poynting's theorem is about power conservation and is derived from Maxwell's equations.

$$\underbrace{-\oint_S \left(\vec{\mathscr{E}} \times \vec{\mathscr{H}}\right)\cdot d\vec{s}}_{\substack{\text{Total power flowing}\\ \text{into the surface } S.}} = \underbrace{\frac{\partial}{\partial t}\int_V \left(W_m + W_e\right) dv}_{\substack{\text{Rate at which the stored}\\ \text{energy in volume } V \text{ is increasing.}}} + \underbrace{\int_V \left(\sigma \vec{\mathscr{E}} \cdot \vec{\mathscr{E}}_e\right) dv}_{\substack{\text{Total power dissipated}\\ \text{in volume } V.}}$$

Units are in watts.

$\vec{\mathscr{S}}$ POYNTING VECTOR [W/m^2]

The Poynting vector is the power density at a point in space, i.e. the power flowing out of a tiny area ds. Units are in watts per meter squared. I haven't found a good font to do cursives with yet; the Poynting vector is supposed to be a cursive capital S.

Instantaneous Poynting vector:

$$\vec{\mathscr{S}}(z,t) = \vec{\mathscr{E}} \times \vec{\mathscr{H}} = \frac{1}{2}\mathscr{Re}\left\{\vec{E}\times\vec{H}^*\right\} + \frac{1}{2}\mathscr{Re}\left\{\left(\vec{E}\times\vec{H}\right)e^{-j2\omega t}\right\}$$

Hint: Convert to the time domain (sine and cosine), then perform $\vec{\mathscr{E}} \times \vec{\mathscr{H}}$.

Time-averaged Poynting vector:

$$\left\langle \vec{\mathscr{S}}(z,t)\right\rangle = \frac{1}{T}\int_0^T \left(\vec{\mathscr{E}} \times \vec{\mathscr{H}}\right) dt = \frac{1}{2}\mathscr{Re}\left\{\vec{E}\times\vec{H}^*\right\}$$

Hint: Either integrate the instantaneous Poynting vector or use the simpler method involving the cross product $\vec{\mathscr{E}} \times \vec{\mathscr{H}}$*. Note that $\vec{\mathscr{H}}$* is just $\vec{\mathscr{H}}$ with the signs reversed on all the **j**s.

$T = \dfrac{2\pi}{\omega} = \dfrac{1}{f}$ the period [s]

$\vec{E}$ = the electric field vector in phasor notation [V/m]

$\vec{H}^*$ = the complex conjugate of the magnetic field intensity in phasor notation [A/m]

$\vec{\mathscr{A}}$ VECTOR MAGNETIC POTENTIAL [Wb/m]

The vector magnetic potential points in the direction of current.

$$\vec{\mathscr{E}} = -j\omega\vec{\mathscr{A}} + \frac{1}{j\omega\mu_0\varepsilon_0}\nabla\left(\nabla\cdot\vec{\mathscr{A}}\right) \qquad \boxed{\nabla\cdot\vec{\mathscr{A}} = \vec{\mathscr{B}}}$$

$$\vec{\mathscr{H}} = \frac{1}{\mu_0}\left(\nabla\times\vec{\mathscr{A}}\right) \qquad \nabla^2\vec{\mathscr{A}} + \underbrace{\omega^2\mu_0\varepsilon_0}_{k_0^2}\vec{\mathscr{A}} = -\mu_0\vec{\mathscr{J}}$$

In Cartesian coordinates:

$$\nabla^2 A_x + k_0^2 A_x = -\mu_0 J_x$$
$$\nabla^2 A_y + k_0^2 A_y = -\mu_0 J_y$$
$$\nabla^2 A_z + k_0^2 A_z = -\mu_0 J_z$$

This statement says that current J_x produces only flux A_x, J_y produces A_y, etc.

VECTOR HELMHOLTZ EQUATION

$$\boxed{\nabla^2\vec{E} + k^2\vec{E} = 0}$$

In the case of a uniform plane wave where E has only an x component and is only a function of z, the equation reduces to

$$\boxed{\frac{d^2E_x}{dz^2} + k^2E_x = 0}$$

Note that this is a second-order ordinary differential equation (the same form as the wave equation) and has the general solution

$$E_x(z) = C_1e^{-jkz} + C_2e^{+jkz}$$

C_1 and C_1 can be determined from boundary conditions. The real or instantaneous electric field can be found as

$$\mathscr{E}_x(z,t) = \mathscr{Re}\left\{\left(C_1e^{-jkz} + C_2e^{+jkz}\right)e^{j\omega t}\right\}$$
$$= C_1\cos(\omega t - kz) + C_2\cos(\omega t + kz)$$

$\vec{E}$ = the electric field vector in phasor notation [V/m]

$k = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi}{\lambda}$, the wave number or propagation constant [m^{-1}]

ω = angular frequency of the source [radians/s]

ε_0 = permittivity of free space 8.85 × 10^{-12} [F/m]

μ_0 = permeability of free space 4π×10^{-7} [H/m]

v_p VELOCITY OF PROPAGATION [m/s]

The velocity of propagation is the speed at which a wave moves through the medium. The velocity approaches the speed of light but may not exceed the speed of light since this is the maximum speed at which information can be transmitted.

$$\boxed{v_p = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{\omega}{k}}$$

If the phase constant k is complex ($k = k' - jk''$), then the relation holds using the real part of the phase constant k': $v_p = \omega/k'$.

ε = permittivity of the material [F/cm]

μ = permeability of the material [H/cm]

ω = frequency [radians/second]

$k = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi}{\lambda}$ phase constant [m^{-1}]

RADIAL WAVES

SENSE, AXIAL RATIO, AND TILT ANGLE

These three parameters determine whether an electromagnetic wave has linear, circular, or elliptical polarization, define which direction the wave rotates in time, and describe the elongation and angular orientation to the x-axis.

Note that the polarization also depends on the angle of observation. For example, moving off axis from a circularly polarized wave causes the polarization to become elliptical, and at 90° off axis, it is linearly polarized.

sense: The rotation of the wave in time as viewed from the receiving antenna. RH or LH. For linearly polarized waves, sense is irrelevant.

axial ratio: The length of the major axis divided by the length of the minor axis. The value can range from 1 (circularly polarized) to infinity (linearly polarized).

tilt angle: The tilting of the major axis with respect to the x-axis. For circular polarization the value is irrelevant.

To determine the three parameters, first determine the direction of propagation, e.g. an e^{-jkz} term indicates propagation in the $+z$ direction due to the negative sign in the exponent. Next, determine the values for E_L and E_R (two new terms defined for this purpose) and convert these complex values to polar notation.

$$E_L = \frac{E_x - \mathrm{j}E_y}{2} = |E_L| e^{+\mathrm{j}\theta_L} \qquad E_R = \frac{E_x + \mathrm{j}E_y}{2} = |E_R| e^{+\mathrm{j}\theta_R}$$

The **sense** is LH if $|E_L| > |E_R|$ and RH if $|E_R| > |E_L|$. If $|E_L| = |E_R|$, then the polarization is linear. **NOTE:** This assumes a wave propagating in the $+z$ direction. For a wave traveling in the $-z$ direction, **reverse the sense** found by this method.

The **axial ratio** is $\text{axial ratio} = \left| \frac{|E_R| + |E_L|}{|E_R| - |E_L|} \right|$

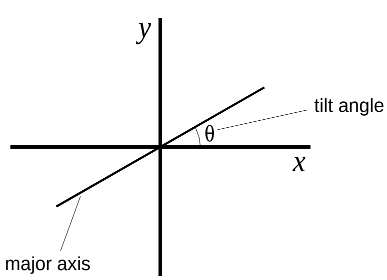
The **tilt angle** is $\text{tilt angle} = \frac{1}{2}\left(\theta_R - \theta_L\right)$

Be careful to preserve the signs of the angles when finding the tilt.

LINEAR POLARIZATION

$$\vec{E} = \left(\hat{x}E_x + \hat{y}E_y\right)e^{-\mathrm{j}kz}$$

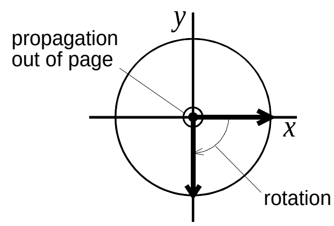
An linearly polarized wave is characterized by E_x and E_y in phase, a finite tilt angle, and an axial ratio of infinity.



CIRCULAR POLARIZATION

A wave is circularly polarized when 1) E_x and E_y are equal in magnitude and 2) they are 90° out of phase as in the example above. In the case of the equation below, propagation is in the $+z$ direction (out of the page) due to e being raised to a negative power.

Example: $\vec{E} = \left(\hat{x}2 - \mathrm{j}\hat{y}2\right)e^{-\mathrm{j}kz}$



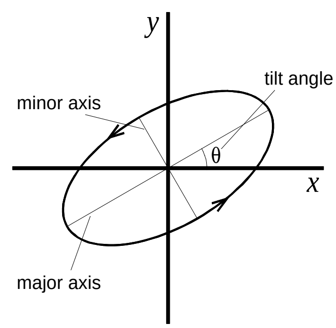
$$\vec{E}\left(z=0\right) = \hat{x}E_x \cos\omega t - \hat{y}E_y \sin\omega t$$

For the case where $z=0$, at $t=0$ the electric field vector points along the $+x$ axis. At $\omega t = \pi/2$, the vector points in the $-y$ direction. To determine sense, point the thumb in the direction of propagation (out of the page in this case) and verify that the fingers curl in the direction of vector rotation. Whichever hand this works with determines the **sense**—right-hand or left-hand. In this case it is the left hand. This method of determining sense is an alternative to the previously mentioned method using the E_R and E_L values.

ELLIPTICAL POLARIZATION

$$\vec{E} = \left(\hat{x}E_x + \hat{y}E_y\right)e^{-\mathrm{j}kz}$$

An elliptically polarized wave is characterized by a finite axial ratio greater than one.



LOSSLESS MATERIALS

Characteristics of plane waves in common lossless materials:

Because $(\mu, \varepsilon) > (\mu_0, \varepsilon_0)$,

$k = \omega\sqrt{\mu\varepsilon}$ gets larger

$v_p = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu\varepsilon}}$ gets smaller

$\lambda = \dfrac{2\pi}{k}$ gets smaller

so the dimensions of an antenna would be smaller…

LOSSY/DISSIPATIVE MATERIALS

Lossy materials can carry some current so this introduces a new term, conductivity σ (sigma) in units of moes per meter [℧/m]. This new term results in complex permittivity. To calculate for lossy materials, simply substitute the new value for complex permittivity into the old equations. For example:

$$\nabla \times \vec{H} = \mathrm{j}\omega\varepsilon\vec{E} \quad \Rightarrow \quad \nabla \times \vec{H} = \mathrm{j}\omega\varepsilon_{\mathrm{new}}\vec{E}$$

e(w) PERMITTIVITY OF THE IONOSPHERE [F/m]

The permittivity of the earth's ionosphere can be described by an **unmagnetized cold plasma** and is dependent on the frequency of the incident radio wave. For the FM band, the permittivity of the ionosphere is about the same as that of free space so the signal passes through it. For the AM band, the permittivity of the ionosphere is much different from free space (it's actually a negative value) so the wave is reflected.

$$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right), \quad \textit{where } \omega_p^2 = \frac{Nq}{m\varepsilon_0}$$

ε_0 = permittivity of free space 8.85 × 10^{-12} [F/m]
ω_p = plasma frequency? [radians/second]
ω = radio frequency [radians/second]
N = electron density, e.g. 10^{12} [m^{-3}]
q = the electron charge, 1.6022×10^{-19} [C]
m = electron mass 9.1094×10^{-31} [kg]

enew COMPLEX PERMITTIVITY [F/m]

Complex permittivity is a characteristic of lossy materials. The imaginary part of $\varepsilon_{\mathrm{new}}$ accounts for heat loss in the medium due to damping of the vibrating dipole moments.

$$\varepsilon_{\mathrm{new}} = \varepsilon\left(1 - \mathrm{j}\frac{\sigma}{\omega\varepsilon}\right) = \varepsilon' - \mathrm{j}\varepsilon'',$$

ε = permittivity [F/m]
σ = (*sigma*) conductivity [℧/m or Siemens/meter]
ε' = the real part of complex permittivity [F/m]
ε'' = the imaginary part of complex permittivity [F/m]

tan q LOSS TANGENT

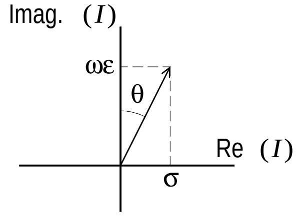
The loss tangent, a value between 0 and 1, is the ratio of conduction current to the displacement current in a lossy medium, or the loss coefficient of a wave after it has traveled one wavelength. This is the way data is usually presented in texts.

$$\boxed{\tan\theta = \frac{\sigma}{\omega\varepsilon}}$$

Graphical representation of loss tangent:

For a dielectric, $\tan\theta \ll 1$.

$$\alpha \approx \frac{1}{2}(\tan\theta)\beta = \frac{\pi}{\lambda}\tan\theta$$



ωε is proportional to the amount of current going through the capacitance C.
σ is proportional to the amount current going through the conductance G.

d SKIN DEPTH [m]

The skin depth or penetration depth is the distance into a material at which a wave is attenuated by 1/e (about 36.8%) of its original intensity.

$$\boxed{\delta = \frac{1}{k''}}$$

where $k = k' - \mathrm{j}k'' = \omega\sqrt{\mu_0 \varepsilon_{\text{new}}}$

and $\varepsilon_{\text{new}} = \varepsilon\left(1 - \mathrm{j}\frac{\sigma}{\omega\varepsilon}\right)$

$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$ phase constant [m^{-1}]

ω = frequency [radians/second]

μ_0 = permeability of free space $4\pi\times10^{-7}$ [H/m]

ε = permittivity [F/m]

σ = (*sigma*) conductivity [℧/m or Siemens/meter]

Skin Depths			
	60 Hz	**1 MHz**	**1 GHz**
silver	8.27 mm	0.064 mm	0.0020 mm
copper	8.53 mm	0.066 mm	0.0021 mm
gold	10.14 mm	0.079 mm	0.0025 mm
aluminum	10.92 mm	0.084 mm	0.0027 mm
iron	0.65 mm	0.005 mm	0.00016 mm

W_m MAGNETIC ENERGY

Energy stored in a magnetic field [Joules].

$$W_m = \frac{1}{2\mu_0}\int_V \mathscr{B}^2 dv'$$

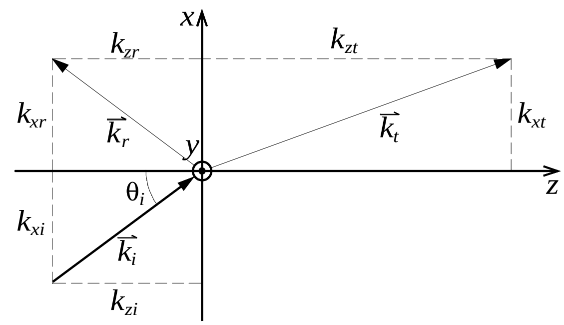
W_m = energy stored in a magnetic field [J]

μ_0 = permeability constant $4\pi\times10^{-7}$ [H/m]

$\mathscr{B}$ = magnetic flux density [Wb/m^2 or T]

REFLECTION AND TRANSMISSION

k_{xi}, etc. WAVE VECTOR COMPONENTS



$$k_i = \omega\sqrt{\mu_1\varepsilon_1} \qquad k_{xi} = \omega\sqrt{\mu_1\varepsilon_1}\sin\theta_i$$

$$k_t = \omega\sqrt{\mu_2\varepsilon_2} \qquad k_{zi} = \omega\sqrt{\mu_1\varepsilon_1}\cos\theta_i$$

μ_1 = permeability of the medium of the incident wave [H/m]

μ_2 = permeability of the medium of the transmitted wave [H/m]

ε_1 = permittivity of the medium of the incident wave [F/m]

ε_2 = permittivity of the medium of the transmitted wave [F/m]

$\vec{k}$ WAVE VECTOR IN 2 DIMENSIONS [m^{-1}]

The vector $\vec{k}$ is in the direction of propagation.

$$\boxed{\vec{k} = k\sin\theta\hat{x} + k\cos\theta\hat{z} = k_x\hat{x} + k_z\hat{z}}$$

Dispersion Relationship: $\boxed{k_x^2 + k_z^2 = \omega^2\mu\varepsilon}$

k_0 PHASE CONSTANT IN FREE SPACE [m^{-1}]

$$\boxed{k_0 = \omega\sqrt{\mu_0\varepsilon_0}}$$

$y_{\perp}, y_{\|}$ PARAMETER OF DISCONTINUITY

The parameter is used in finding the reflection and transmission coefficients.

$$\boxed{y_{\perp} = \frac{\mu_1}{\mu_2}\frac{k_{zt}}{k_{zi}}} \quad \boxed{y_{\|} = \frac{\varepsilon_1}{\varepsilon_2}\frac{k_{zt}}{k_{zi}}}$$, where

$$k_{xt}^2 + k_{zt}^2 = \omega^2\mu_2\varepsilon_2 \quad \text{and} \quad k_{xt} = k_{xi}$$

μ_1 = permeability of the medium of the incident wave [H/m]
μ_2 = permeability of the medium of the transmitted wave [H/m]
ε_1 = permittivity of the medium of the incident wave [F/m]
ε_2 = permittivity of the medium of the transmitted wave [F/m]
k_{zt} = z-component of the phase constant of the transmitted wave [m^{-1}]
k_{zi} = z-component of the phase constant of the incident wave [m^{-1}]

$\tau_{\perp}, \tau_{\|}$ TRANSMISSION COEFFICIENT

The coefficient used in determining the amplitude of the transmitted wave.

$$\boxed{\tau_{\perp} = \frac{2}{1+y_{\perp}}} \quad \boxed{\tau_{\|} = \frac{2}{1+y_{\|}}}$$

$y_{\perp}$ = parameter of discontinuity for perpendicular polarized waves
$y_{\|}$ = parameter of discontinuity for parallel polarized waves

The reflection and transmission coefficients are related.

$$1+\Gamma_{\perp} = \tau_{\perp}$$

$\Gamma_{\perp}, \Gamma_{\|}$ REFLECTION COEFFICIENT

This value, when multiplied by the incident wave E_y and reversing the sign of the wave component perpendicular to the plane of discontinuity, yields the reflected wave. For the reflection coefficient of a transmission line, see p17.

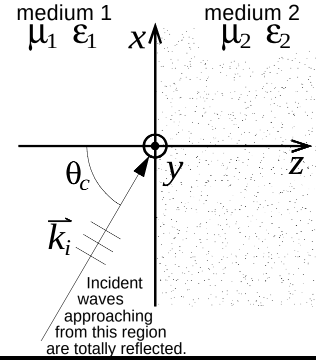
$$\boxed{\Gamma_{\perp} = \frac{1-y_{\perp}}{1+y_{\perp}}} \quad \boxed{\Gamma_{\|} = \frac{1-y_{\|}}{1+y_{\|}}}$$

$y_{\perp}$ = parameter of discontinuity for perpendicular polarized waves
$y_{\|}$ = parameter of discontinuity for parallel polarized waves

θ_c CRITICAL ANGLE

The minimum angle of incidence for which there is total reflection. A critical angle exists for waves in a dense material encountering a less dense material, i.e. $\varepsilon_2 < \varepsilon_1$. Applies to both perpendicular and parallel polarization.

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}}$$



BOUNDARY CONDITIONS

The tangential components of the electromagnetic wave are equal across the boundary (discontinuity) in materials with finite σ (most materials—perfect conductors are the exception). Remember that the term *boundary* means that we're talking about the case where $z = 0$. The tangential components are the x and y components, so the boundary conditions are:

$\perp$ Polarization: $\boxed{E_y^i + E_y^r = E_y^t}$ and $\boxed{H_x^i + H_x^r = H_x^t}$

$\|$ Polarization: $\boxed{E_x^i + E_x^r = E_x^t}$ and $\boxed{H_y^i + H_y^r = H_y^t}$

The only difference for a **perfect conductor** is that the sum of the incident and reflected magnetic components is not equal to the transmitted component (zero) but to the surface current density J_S in units of A/m.

$\perp$ Pol.: $\boxed{H_x^i + H_x^r = J_S}$ and $\|$ Pol.: $\boxed{H_y^i + H_y^r = J_S}$

The **general expressions for boundary conditions** are

$$\boxed{\hat{n}\times\left(\vec{E}_1 - \vec{E}_2\right) = 0} \quad \text{and} \quad \boxed{\hat{n}\times\left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S}$$

where $\hat{n}$ is a unit vector normal to the plane of discontinuity ($\hat{n} = -\hat{z}$ in the examples here). This also gives us the direction of current J_S. These two equations apply in all situations with the understanding that $J_S = 0$ for all materials except perfect conductors and that $E_2 = H_2 = 0$ for perfect conductors (due to total reflection).

SNELL'S LAW

Relates the angles of incidence and transmission to the properties of the materials. Don't use Snell's law with lossy materials.

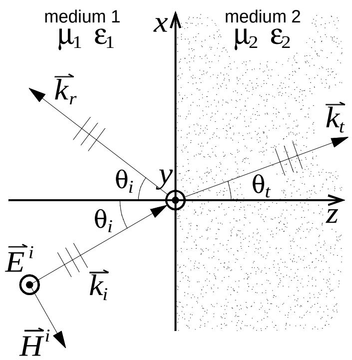
$\boxed{k_i \sin\theta_i = k_t \sin\theta_t}$ or $\boxed{\sqrt{\mu_i\varepsilon_i}\,\sin\theta_i = \sqrt{\mu_t\varepsilon_t}\,\sin\theta_t}$

k_i = phase constant of the incident wave [m^{-1}]
k_t = phase constant of the transmitted wave [m^{-1}]
θ_i = angle of incidence [°]
θ_t = angle of reflection [°]

^ PERPENDICULAR POLARIZATION

PERPENDICULAR POLARIZATION
Wave Reflection/Transmission

$\vec{E}^i$ is perpendicular to the plane of incidence (the xz plane) with components E_y and H_x, propagating in the $\vec{k}_i$ direction. The wave encounters a discontinuity at the xy plane with a reflection in the $\vec{k}_r$ direction and a transmitted wave in the $\vec{k}_t$ direction. If the wave is entering a denser medium ($\varepsilon_{r2} > \varepsilon_{r1}$) then the transmitted wave will bend toward the z-axis.



μ_i = permeability of the medium of the incident wave [H/m]
μ_t = permeability of the medium of the transmitted wave [H/m]
ε_i = permittivity of the medium of the incident wave [F/m]
ε_t = permittivity of the medium of the transmitted wave [F/m]
θ_i = angle of incidence [°]
θ_t = angle of reflection [°]
$\vec{k}_i, \vec{k}_r, \vec{k}_t$ = wave vectors for the incident, reflected, and transmitted plane waves [rad./m]

PERPENDICULAR POLARIZATION EQUATIONS
for incident, reflected, and transmitted waves

The form of the expressions for the electrical and magnetic components of the perpendicularly polarized wave encountering a discontinuity. Note the placement of the k_{zi} and k_{xi} terms in the expression for the magnetic field due to it's perpendicular orientation to the electric field.

Incident: $\boxed{\vec{E}^i = \hat{y}E_0 e^{-jk_{xi}x} e^{-jk_{zi}z}}$

$$\boxed{\vec{H}^i = \frac{E_0}{\omega\mu_i}\left(-\hat{x}\,k_{zi} + \hat{z}\,k_{xi}\right) e^{-jk_{xi}x} e^{-jk_{zi}z}}$$

Reflected: $\boxed{\vec{E}^r = \hat{y}E_0\Gamma_\perp e^{-jk_{xr}x} e^{+jk_{zr}z}}$

$$\boxed{\vec{H}^r = \frac{E_0\Gamma_\perp}{\omega\mu_i}\left(+\hat{x}\,k_{zr} + \hat{z}\,k_{xr}\right) e^{-jk_{xr}x} e^{+jk_{zr}z}}$$

Transmitted: $\boxed{\vec{E}^t = \hat{y}E_0\tau_\perp e^{-jk_{xt}x} e^{-jk_{zt}z}}$

$$\boxed{\vec{H}^t = \frac{E_0\tau_\perp}{\omega\mu_t}\left(-\hat{x}\,k_{zt} + \hat{z}\,k_{xt}\right) e^{-jk_{xt}x} e^{-jk_{zt}z}}$$

where:

The x-components of the phase constant are equal for the incident, reflected, and transmitted waves. This is called the *phase matching condition* and is determined by the boundary conditions.

$\boxed{k_{xi} = k_{xr} = k_{xt}}$ phase matching condition

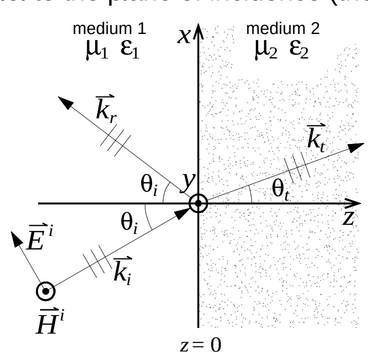
Other relations: $\boxed{k_{zi} = k_{zr}}$ and $\boxed{k_{xt}^2 + k_{zt}^2 = \omega^2\mu_t\varepsilon_t}$

Also see **Characteristic Impedance** (p5) and **Wave Vector Components** (p13) for help with these problems.

|| PARALLEL POLARIZATION

PARALLEL POLARIZATION

$\vec{E}^i$ is parallel to the plane of incidence (the xz plane).



PARALLEL POLARIZATION EQUATIONS
for incident, reflected, and transmitted waves

The form of the expressions for the electrical and magnetic components of the parallel polarized wave encountering a discontinuity.

Incident: $$\boxed{\vec{E}^{i} = \frac{1}{\omega\varepsilon_1}\left(k_{zi}\hat{x} - k_{xi}\hat{z}\right) H_0 e^{-\mathrm{j}k_{xi}x} e^{-\mathrm{j}k_{zi}z}}$$

$$\boxed{\vec{H}^{i} = \hat{y} H_0 e^{-\mathrm{j}k_{xi}x} e^{-\mathrm{j}k_{zi}z}}$$

Reflected: $$\boxed{\vec{E}^{r} = \frac{1}{\omega\varepsilon_1}\left(-k_{zr}\hat{x} - k_{xr}\hat{z}\right) H_0 \Gamma_{\|} e^{-\mathrm{j}k_{xr}x} e^{+\mathrm{j}k_{zr}z}}$$

$$\boxed{\vec{H}^{r} = \hat{y} H_0 \Gamma_{\|} e^{-\mathrm{j}k_{xr}x} e^{+\mathrm{j}k_{zr}z}}$$

Transmitted: $$\boxed{\vec{E}^{t} = \frac{1}{\omega\varepsilon_2}\left(k_{zt}\hat{x} - k_{xt}\hat{z}\right) H_0 \tau_{\|} e^{-\mathrm{j}k_{xt}x} e^{-\mathrm{j}k_{zt}z}}$$

$$\boxed{\vec{H}^{t} = \hat{y} H_0 \tau_{\|} e^{-\mathrm{j}k_{xt}x} e^{-\mathrm{j}k_{zt}z}}$$

where:

The x-components of the phase constant are equal for the incident, reflected, and transmitted waves. This is called the *phase matching condition* and is determined by the boundary conditions.

$\boxed{k_{xi} = k_{xr} = k_{xt}}$ phase matching condition

Other relations: $\boxed{k_{zi} = k_{zr}}$ and $\boxed{k_{xt}^2 + k_{zt}^2 = \omega^2\mu_t\varepsilon_t}$

Also see **Characteristic Impedance** (p5) and **Wave Vector Components** (p13) for help with these problems.

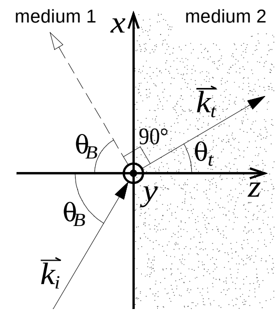
q_B BREWSTER ANGLE

Named for Scottish physicist, Sir David Brewster, who first proposed it in 1811. For electromagnetic waves, the Brewster angle **applies only to parallel polarization**. It is the angle of incidence at which there is total transmission of the incident wave, i.e. $\Gamma_{\|} = 0$.

For light waves it is the angle of incidence that results in an angle of 90° between the transmitted and reflected waves. Also called the ***polarizing angle***, this results in the reflected wave being polarized, with vibrations perpendicular to the ***plane of incidence*** (in other words, perpendicular to the page).

In acoustic applications such as ***lithotripsy***, θ_B is called the ***angle of intromission***, used for blasting kidney stones.

$$\boxed{\tan\theta_B = \frac{k_t}{k_i} = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}}$$



s LOSSY MEDIUM

When a lossy medium is involved, use the same equations but replace ε with ε_{new}. The imaginary part of ε_{new} accounts for heat loss in the medium due to damping of the vibrating dipole moments. Don't use Snell's law with lossy mediums.

k_{zt} must have the form k_{zt}'-jk_{zt}'', positive real and negative imaginary, i.e. change it if you have to. The solution will contain the term

$$e^{-\mathrm{j}k_{xi}x} e^{-\mathrm{j}k_{zt}'z} e^{-k_{zt}''z}$$

Medium 1	Medium 2
$\mu_1\ \varepsilon_1$	$\mu_2\ \underbrace{\varepsilon_2\ \sigma_2}_{\varepsilon_{new} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}$

s ® ¥ PERFECT CONDUCTOR

Everything gets reflected. Since we don't know the surface current density J_s, only one boundary condition is useful. This is sufficient since we know the transmitted waves are zero. Remember that the term *boundary* means $z = 0$.

The tangential components of the incident and reflected electric fields are out of phase at the boundary so that they cancel.

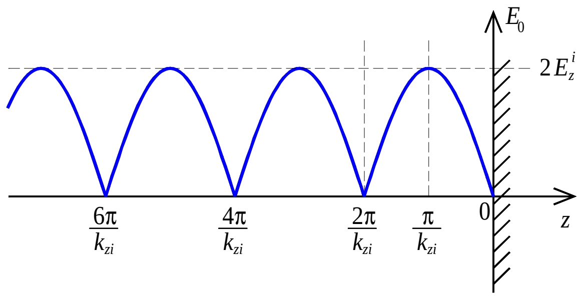
The tangential components of the incident and reflected magnetic fields are in phase at the boundary creating a strong magnetic field that produces surface current J_s [A/m].

Boundary Conditions: $$\boxed{\begin{aligned}\hat{n}\times\vec{H}_1 &= \vec{J}_S\\ \hat{n}\times\vec{E}_1 &= 0\end{aligned}}$$

where $\hat{n} = -\hat{z}$

^ Polarized: $E_y^i + E_y^r = 0$ $\boxed{\Gamma_\perp = -1}$

$$E_0 e^{-jk_{xi}x} + E_0\Gamma_\perp e^{-jk_{xi}x} = 0$$

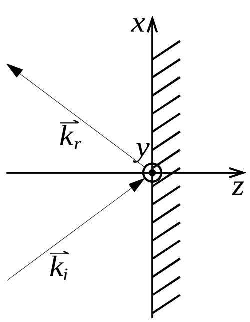


The z component of the incident and reflected electric fields of the ⊥ polarized wave produce a standing wave with constructive peaks spaced at $2\pi/k_{zi}$ apart, beginning π/k_{zi} from the conductor surface.

|| Polarized: $E_x^i + E_x^r = 0$ $\boxed{\Gamma_\| = +1}$

$$\frac{1}{\omega\varepsilon_1}k_{zi}H_0 e^{-jk_{xi}x} - \frac{1}{\omega\varepsilon_1}k_{zi}H_0\Gamma_\| e^{-jk_{xi}x} = 0$$

$$\vec{E}^t = \vec{H}^t = 0$$
$$\nabla\cdot\vec{J}_S + j\omega\rho_S = 0$$
$$\boxed{\vec{J}_S = \hat{n}\times\vec{H}_{(z=0)}}$$



where $\hat{n} = -\hat{z}$ in this example, and J_S is the surface current [A/m]. Surface current is present only in a perfect conductor (see Boundary Conditions p14). *Surface* means $z = 0$.

ANTENNAS

G REFLECTION COEFFICIENT [unitless]

This is the reflection coefficient for the transmission line. For reflected waves in space, see p14.

$$\boxed{\Gamma = \rho e^{j\theta} = \frac{Z_L - Z_0}{Z_L + Z_0}} \text{ and } Z_L = Z_0\frac{1+\Gamma}{1-\Gamma}$$

ρ = magnitude of the reflection coefficient [no units]
θ = phase angle of the reflection coefficient [no units]
Z_L = load (antenna) impedance [Ω]
Z_0 = transmission line (characteristic) impedance [Ω]

SWR STANDING WAVE RATIO [V/V]

Also called the *voltage standing wave ratio* or VSWR.

$$\boxed{\text{SWR} = \frac{1+\rho}{1-\rho}}$$

ρ = magnitude of the reflection coefficient [no units]

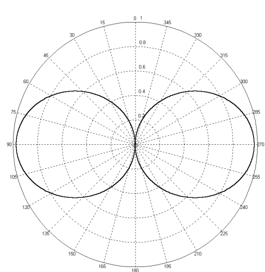
RADIATION PATTERN

The radiation pattern of an antenna is the relative strength of the absolute value of the electric field as a function of θ and φ. The radiation pattern is the same for receiving antennas as for transmitting antennas.

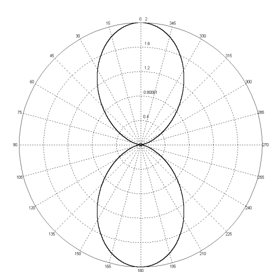
Radiation Pattern: $\left|\vec{\mathscr{E}}(\theta,\phi)\right|$

The example on the left below is the radiation pattern for a single dipole oriented along the verticle axis. The pattern on the right is for a 2-dipole array with the elements lying in the horizontal plane, one to the left and one to the right of center, with their lengths extending into the page.

Radiation pattern $|\mathscr{E}$(q)| for a single dipole



Radiation pattern $|\mathscr{E}$(90,f)| for 2-element dipole with d=l/2



POWER

Time-Averaged Poynting Vector [W/m²]

$$\boxed{\left\langle \vec{\mathscr{S}} \right\rangle = \tfrac{1}{2}\mathscr{Re}\left\{\vec{\mathscr{E}}^{\text{ ff}} \times \vec{\mathscr{H}}^{\text{ ff}*}\right\}}$$

Total Time-Averaged Radiated Power [W]

$$\boxed{\left\langle P_{\text{rad}} \right\rangle = \int_S \left\langle \vec{\mathscr{S}} \right\rangle \cdot d\vec{s} = \oint_S \left\langle \vec{\mathscr{S}} \right\rangle \cdot \hat{r}\left(r^2 \sin\theta \, d\theta \, d\phi\right)}$$

dBm DECIBELS RELATIVE TO 1 mW

The decibel expression for power. The logarithmic nature of decibel units translates the multiplication and division associated with gains and losses into addition and subtraction.

0 dBm = 1 mW
20 dBm = 100 mW
-20 dBm = 0.01 mW

$$P(\text{dBm}) = 10\log\left[P(\text{mW})\right]$$

$$P(\text{mW}) = 10^{P(\text{dBm})/10}$$

GAIN

$$\text{Gain} = \text{Directivity} \times \text{Efficiency}$$

EFFICIENCY

$$\text{Efficiency} = \frac{\text{Power radiated}}{\text{Power radiated} + \text{Power dissipated in conductor}}$$

(DIR) DIRECTIVITY [no units/dB]

The directivity is the gain in the direction of maximum radiation compared to an omnidirectional spherical wave.

$$\boxed{\text{Directivity} = \frac{\left\langle \vec{\mathscr{S}} \right\rangle_{\max}}{\left\langle P_{\text{rad}} \right\rangle / \left(4\pi r^2\right)}} \quad \text{[no units]}$$

where $\left\langle \vec{\mathscr{S}} \right\rangle = \tfrac{1}{2}\mathscr{Re}\left\{\vec{\mathscr{E}}^{\text{ ff}} \times \vec{\mathscr{H}}^{\text{ ff}*}\right\} = \dfrac{\left|\vec{\mathscr{E}}^{\text{ ff}}\right|^2}{2\eta_0}$

and $\left\langle P_{\text{rad}} \right\rangle = \oint_S \left\langle \vec{\mathscr{S}} \right\rangle \cdot \hat{r}\left(r^2 \sin\theta \, d\theta \, d\phi\right)$

Directivity is often expressed in dB:

$$\boxed{(\text{DIR})^{\text{dB}} = 10\log(\text{DIR})}$$

Directivity of Antenna Types

Type	Directivity	dB
Isotropic	1.0	0
Infinitesimal dipole	1.5	1.76
Half-wave dipole	1.64	2.15
Wire-type	~10	~10
Horn	~100	~20
Reflector	10^3-10^9	30-90

FAR FIELD APPROXIMATION

In general, we are only interested in the electric and magnetic fields distant from the antenna. This allows us to simplify the calculations by dropping the *near field* components. As a rule of thumb, the far field region is defined as:

$$r > \frac{2D^2}{\lambda}$$

where D is the diameter or size of the antenna

ISOTROPIC ANTENNA

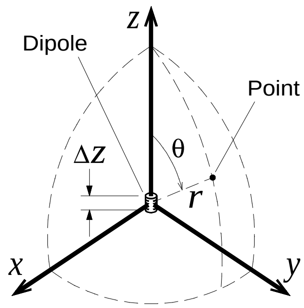
An isotropic antenna is a theoretical antenna that radiates equally in all directions. On any given "spherical shell" in the far field, $\mathscr{E}^{\text{ff}}$ and $\mathscr{H}^{\text{ff}}$ are in phase and are equal in magnitude.

$$\left\langle \mathscr{S} \right\rangle = \frac{\left\langle P_{\text{rad}} \right\rangle}{4\pi r^2} \qquad (\text{DIR}) = 1$$

INFINITESIMAL DIPOLE AT ORIGIN

A theoretical model of a very short dipole antenna, the most basic of antennas. A points in the direction of current, in this case z. Oscillation occurs along the z-axis.

Vector magnetic potential and magnetic field at a point in space due to an infinitesimal dipole antenna at the origin.



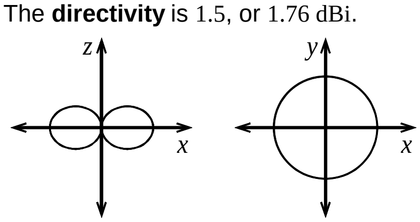
$$\vec{\mathscr{A}} = \hat{z}\mu_0\left(I\Delta z\right)\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}$$

$$\vec{\mathscr{H}} = \frac{1}{\mu_0}\left(\nabla\times\vec{\mathscr{A}}\right) = \left(I\Delta z\right)\nabla\times\left(\hat{z}\,\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}\right)$$

In the **far field**, only the slowest-decaying components are significant.

$$\boxed{\begin{aligned}\vec{\mathscr{H}}^{\,\mathrm{ff}} &= \hat{\phi}\,\mathrm{j}k_0\left(I\Delta z\right)\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}\sin\theta\\ \vec{\mathscr{E}}^{\,\mathrm{ff}} &= \hat{\theta}\,\mathrm{j}k_0\eta_0\left(I\Delta z\right)\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}\sin\theta\end{aligned}}$$

The **radiation pattern** of the infinitesimal dipole is non-isotropic. The **directivity** is 1.5, or 1.76 dBi.



Time-Averaged Poynting Vector [W/m^2]

$$\boxed{\left\langle\vec{\mathscr{S}}\right\rangle = \tfrac{1}{2}\,\mathscr{Re}\left\{\vec{\mathscr{E}}^{\,\mathrm{ff}}\times\vec{\mathscr{H}}^{\,\mathrm{ff}\,*}\right\} = \hat{r}\,\frac{k_0^2\eta_0}{2}\left(I\Delta z\right)^2\frac{\sin^2\theta}{\left(4\pi r\right)^2}}$$

Total Time-Averaged Radiated Power [W]

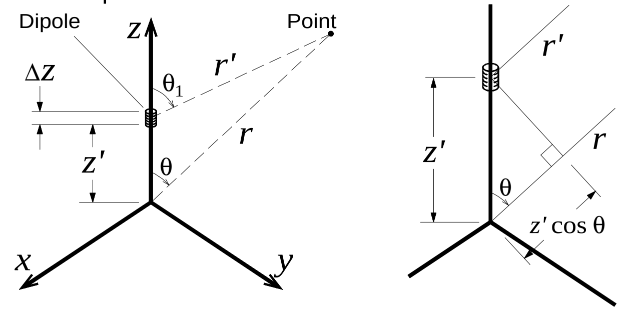
$$\boxed{\left\langle P_{\mathrm{rad}}\right\rangle = \int_S\left\langle\vec{\mathscr{S}}\right\rangle\cdot d\vec{s} = \frac{k_0^2\eta_0\left(I\Delta z\right)^2}{12\pi}}$$

Directivity: $\left(\mathrm{DIR}\right) = 1.5$

$\mathscr{A}$ = vector magnetic potential [Wb/m]
$\mathscr{H}$ = magnetic field intensity [A/m]
r = radial distance from the origin [m]

INFINITESIMAL DIPOLE ON *Z*-AXIS

The infinitesimal dipole at the origin is shifted to another point on the z-axis.



In the **far field**, we can make these approximations.

$$r_1 \approx r - z'\cos\theta \qquad \theta_1 \approx \theta \qquad \hat{\theta}_1 \approx \hat{\theta}$$

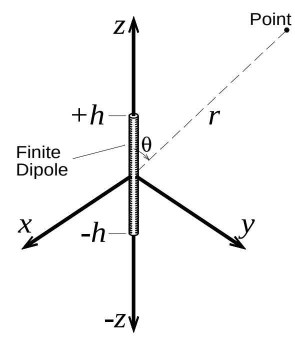
From the expression for electric field of the dipole at the origin, we get.

$$\vec{\mathscr{E}}^{\,\mathrm{ff}} = \hat{\theta}\,\mathrm{j}k_0\eta_0\left(I\nabla z\right)\frac{e^{-\mathrm{j}k_0\left(r-z'\cos\theta\right)}}{4\pi(r\ \underbrace{-z'\cos\theta}_{\text{This term has less effect here than when it appears in the exponent above.}})}\sin\theta$$

FINITE DIPOLE ON *Z*-AXIS

The finite dipole on the z-axis is calculated by summing the infinitesimal dipoles over a finite length.

$I(z')$ is a function describing the current along the dipole. For a ½ -wave dipole centered at the origin, this would be $I(z') = I_0\cos(z'k_0)$.



From superposition

$$\vec{\mathscr{E}}^{\,\mathrm{ff}} = \lim_{\Delta z\to 0}\sum\vec{\mathscr{E}}^{\,\mathrm{ff}} = \int_{-h}^{+h}d\vec{\mathscr{E}}^{\,\mathrm{ff}}$$

$$= \hat{\theta}\int_{-h}^{+h}\mathrm{j}k_0\eta_0\,\frac{I\left(z'\right)dz'e^{-\mathrm{j}k_0 r}e^{+\mathrm{j}k_0 z'\cos\theta}}{4\pi r}\sin\theta$$

And finally the radiation due to an arbitrary line current $I(z')$. We must understand this.

$$\boxed{\vec{\mathscr{E}}^{\,\mathrm{ff}} = \hat{\theta}\,\mathrm{j}k_0\eta_0\,\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}\sin\theta\int_{-h}^{+h}I\left(z'\right)e^{+\mathrm{j}k_0 z'\cos\theta}dz'}$$

$$\boxed{\vec{\mathscr{H}}^{\,\mathrm{ff}} = \frac{1}{\eta_0}\left(\hat{r}\times\vec{\mathscr{E}}^{\,\mathrm{ff}}\right)}$$

See the next section also.

$\mho(q)$ SPACE FACTOR

From the previous section, here is the far-field radiation due to an arbitrary line current:

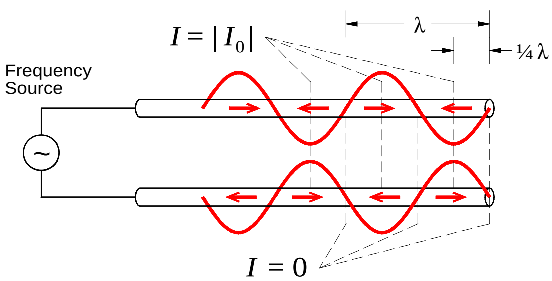
$$\vec{\mathscr{E}}^{\,\text{ff}} = \underbrace{\hat{\theta}\, \mathrm{j} k_0 \eta_0 \frac{e^{-\mathrm{j}k_0 r}}{4\pi r} \sin\theta}_{\substack{\text{ELEMENT PATTERN} \\ \text{the electric field due to} \\ \text{an infinitesimal dipole}}} \underbrace{\int_{-h}^{+h} I(z')\, e^{+\mathrm{j}k_0 z' \cos\theta} dz'}_{\substack{\text{SPACE FACTOR} \\ \mho(\theta)}}$$

The Space Factor depends on the length of the element and the **current distribution** within the element. Note that the space factor resembles a **Fourier transform** of the current distribution, the only difference being the sign of the exponent. Actually it is the Fourier transform since the sign is arbitrary as long as the opposite sign is used in the exponent of the compatible inverse Fourier transform. What this means is that an even distribution of current will result in a more directional radiation pattern and a tapered (Gaussian) current distribution will result in a broader radiation pattern with lower sidelobes. See also Fourier Transform p30 and Fourier Transform Examples p30.

HALF-WAVE DIPOLE

The **half-wave dipole** or **resonant dipole** is the most commonly used antenna primarily because of its impedance, which is easily matched.

To understand the half-wave dipole, first consider the current on a transmission line. The current on one conductor is out of phase with current on the other so that radiation effects are canceled. At ¼ -wavelength from the end, where the line will be bent to form the ½ -wave dipole, current is at a maximum.



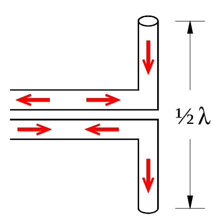
As a result of bending, current is now in phase so that radiation takes place.

Current in the dipole

$$I(z') = I_0 \cos(k_0 z')$$

Impedance: $z_{in} = 73 + j0\ \Omega$

Directivity: 1.64 or 2.15 dB

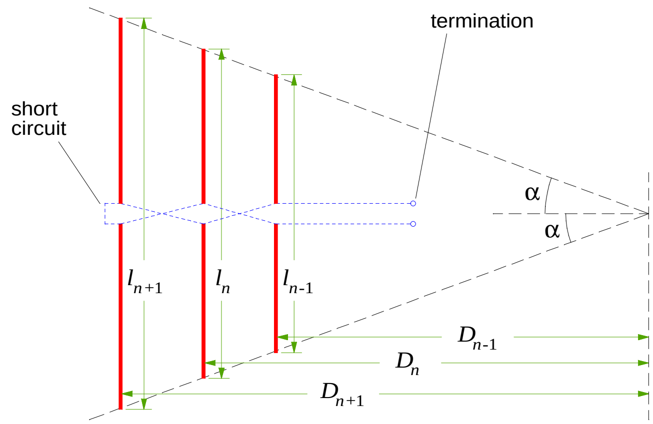


$$\boxed{\vec{\mathscr{E}}^{\,\text{ff}} = \hat{\theta}\, \mathrm{j} k_0 \eta_0 \frac{e^{-\mathrm{j}k_0 r}}{4\pi r} \sin\theta \frac{2I_0 \cos\left(\frac{\pi}{2}\cos\theta\right)}{k_0 \sin^2\theta}}$$

Increasing the thickness of the elements has the effect of broadening the antenna bandwidth.

LOG-PERIODIC ANTENNA

The log-periodic dipole array (LPDA) consists of an array of half-wave dipoles of frequencies f, rf, $r^2 f$, . . ., $r^n f$ which, when plotted on a log scale appear equally spaced. This produces a broadband antenna.



LOG-PERIODIC FREQUENCIES

The relationship between the bandwidth, the number of elements, and the scaling factor for a log-periodic antenna follows. These are my own observations; textbook information is in the next box.

$$\log \frac{f_u}{f_l} = N \log \frac{1}{s_f}$$

The crossover points between bands is then:

$$f_l \left(\frac{1}{s_f}\right)^0,\ f_l \left(\frac{1}{s_f}\right)^1,\ f_l \left(\frac{1}{s_f}\right)^2,\ \cdots\ f_l \left(\frac{1}{s_f}\right)^N$$

The frequencies used to determine the length of each half-wavelength element are the center frequencies of each of the N bands:

$$f_l \left(\frac{1}{s_f}\right)^{0.5},\ f_l \left(\frac{1}{s_f}\right)^{1.5},\ f_l \left(\frac{1}{s_f}\right)^{2.5},\ \cdots\ f_l \left(\frac{1}{s_f}\right)^{N-0.5}$$

f_u = upper bandwidth cutoff frequency [Hz]
f_l = lower bandwidth cutoff frequency [Hz]
N = number of antenna elements
s_f = scaling factor [no units]

LOG-PERIODIC ANTENNA PROPERTIES

Alignment angle $\tan\alpha = \frac{l_n}{2D_n}$

Scaling factor $s_f = \frac{l_n}{l_{n+1}} = \frac{D_n}{D_{n+1}}$, a constant < 1

Spacing parameter $S_p = \frac{d_n}{2l_n} = \frac{d_n}{\lambda} = \frac{1-s_f}{4}\cot\alpha$,

where $d_n = D_{n+1} - D_n$, varies with the element.

and the optimum spacing parameter is

$S_p = -5.76909996\times10^{-4}\times g^2 + 0.0167208176g + 0.0602945516$

where g is the gain in dB,

and the optimum scaling factor is

$s_f = 3.9866009\,S_p + 0.236230336$

Impedance $Z_a = \frac{Z_0^2}{8Z_1}\sqrt{s_f + \frac{8Z_1}{Z_0}}$

where $Z_1 = 276\log\underbrace{\left(\frac{l}{d_0}\right)}_{\text{Element length to diameter ratio}} - 270$ and

Z_0 is the characteristic impedance of the transmission line. The antenna impedance is matched to the line by adjusting the length to diameter ratio.

Design bandwidth $B_s = B_w\left[1.1 + 7.7\left(1-s_f\right)^2\right]\cot\alpha$

where B_w is the desired bandwidth as the ratio of highest to lowest frequency. The design bandwidth is greater than the desired bandwidth.

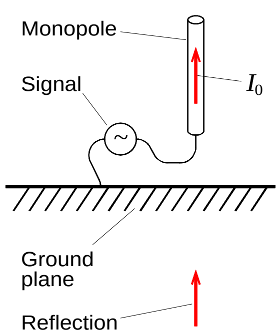
Number of elements $N = \left[1 + \frac{\ln B_s}{\ln\left(1/S_f\right)}\right]$

¼ -WAVE MONOPOLE

A ¼ -wave monopole antenna is essentially half of a ½ -wave dipole antenna.

According to *image theory*, when the monopole is mounted perpendicular to a ground plane, it can be modeled as having a reflected current opposite the plane with flow in the same direction as current in the monopole.

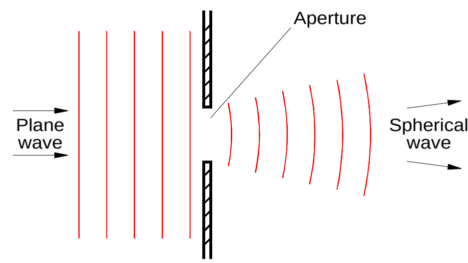
$$Z_{\text{in}} = \frac{V_0/2}{I_0} = 36.5\Omega$$



APERTURE ANTENNAS

APERTURE THEORY

If a plane wave eminates from a source of finite dimensions, the propagating wave assumes spherical characteristics within its radiation pattern. The beamwidth is inversely related to the size (in wavelengths) of the aperture, that is, a large aperture produces a more directional beam. However, sidelobe level is independent of size; it depends on amplitude taper. For a large aperture, the sine θ term in the element pattern disappears since the pattern becomes highly directional with θ ≈ 90°, sinθ ≈ 1.



The **far field** radiation is proportional to the Fourier transform of the current distribution.

Array Factor:

Azimuth: $\text{AF}_x = \left|\frac{\sin\left(ka_x\phi\right)}{ka_x\phi}\right|$

Elevation: $\text{AF}_y = \left|\frac{\sin\left(ka_y\theta\right)}{ka_y\theta}\right|$

$k = 2\pi/\lambda$, phase constant of the carrier wave [m^{-1}]

$2a_x$ = width of aperture [m]

$2a_y$ = height of aperture [m]

ϕ = azimuth angle [radians]

θ = elevation angle [radians]

TOTAL FIELD for apertures and arrays

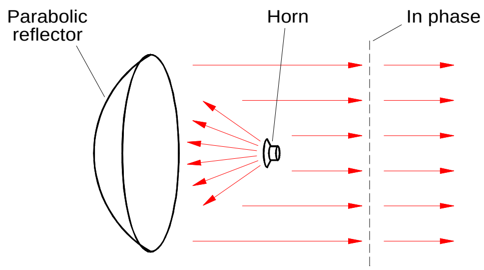
The total radiated field is a product of the element pattern and the array factor.

$$\boxed{\text{Total Field} = \left(\begin{matrix}\text{Element}\\\text{pattern}\end{matrix}\right)\times\left(\begin{matrix}\text{Array}\\\text{factor}\end{matrix}\right)}$$

The **far field** radiation is proportional to the Fourier transform of the current distribution.

CIRCULAR APERTURE ANTENNA

The circular aperture antenna is parabolic in shape so that the wave emitted from the horn is reflected in phase as a plane wave.



For **uniform amplitude** across the beam:

$$\boxed{\text{Beamwidth} = \frac{1 \text{ radian}}{D/\lambda} \approx \frac{60°}{D/\lambda}}$$ **VERY IMPORTANT**

$$\text{Directivity} = \frac{4\pi(\text{Area})}{\lambda^2}$$

For a **tapered aperture** (amplitude varies across the aperture:

$$\text{Directivity} = \frac{4\pi(\text{Area})}{\lambda^2}\eta_a$$

$$\text{Edge taper} = 20\log\frac{J_{\text{edge}}}{J_{\text{center}}}$$

η_a = aperature efficiency, a value less than or equal to 1
D = aperature diameter [m]
J = current density [A/m^2]
λ = wavelength [meters]

Sidelobe level is dependent on amplitude taper, not size:

Sidelobe Levels

Edge Taper [dB]	Beamwidth [°]	Sidelobe [dB]	Aperture Efficiency
-8	65.3°/(D/λ)	-24.7	0.918
-10	67.0°/(D/λ)	-27.0	0.877
-12	68.8°/(D/λ)	-29.5	0.834
-14	70.5°/(D/λ)	-31.7	0.792
-16	72.2°/(D/λ)	-33.5	0.754
-18	73.9°/(D/λ)	-34.5	0.719
-20	75.6°/(D/λ)	-34.7	0.690

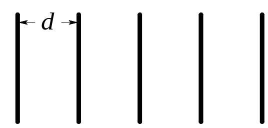
ANTENNA ARRAYS

When two or more antennas are used together, the combination is called an array. A planar arrangement of closely spaced antenna elements essentially produces an aperture of equivalent area.

AF ARRAY FACTOR

The Array Factor is the far-field radiation intensity of the elements of an array, assuming the elements to be isotropic radiators. That is, it contains no information about the type of element used in the array.

AF ARRAY FACTOR
for a linear series of elements

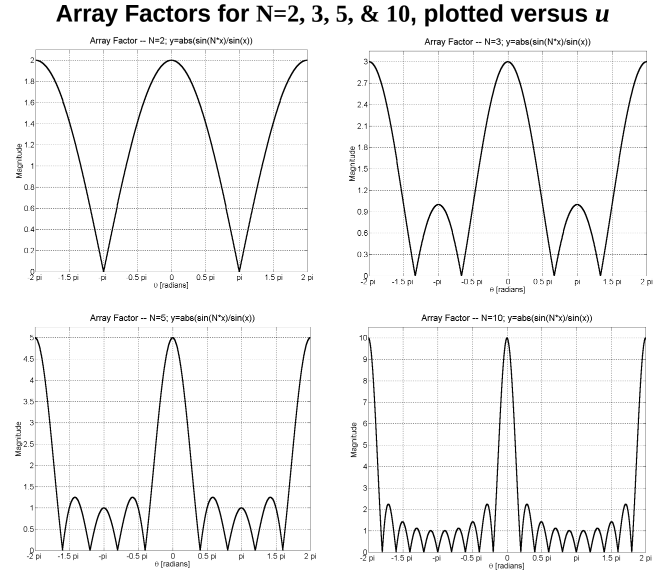


$$\boxed{|\text{AF}| = \left|\frac{\sin(Nu/2)}{\sin(u/2)}\right|, \qquad u = k_0 d \sin\theta \sin\phi + \psi}$$

N = number of array elements [no units]
d = spacing between adjacent elements [m]
θ = angle of elevation. This can be taken as 90° for the x-y plane [radians]
ψ = phase angle between adjacent elements (The direction of the antenna array can be electronically steered by driving the elements at different phases, i.e. progressively altering the phase of each element by the angle ψ with respect to the previous element.) [radians]

|AF| is periodic with period 2π. There are N-2 sidelobes in the AF between 0 and 2π.

Array Factors for N=2, 3, 5, & 10, plotted versus u



ϕ_0 ARRAY STEERING [radians]

The direction of the main lobe of the antenna array can be electronically steered away from the perpendicular by an angle of ϕ by driving the elements at different phases, that is, progressively altering the phase of each element by the angle ψ with respect to the previous element.

$$\boxed{\phi_0 = \sin^{-1}\left(-\frac{\psi}{k_0 d}\right)}$$

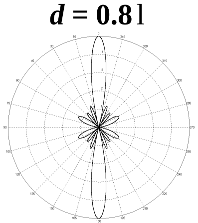
ψ = phase angle of the driving signal between adjacent elements [radians]
k_0 = phase constant in free space [m^{-1}]
d = the distance between adjacent elements [m]

GRATING LOBES

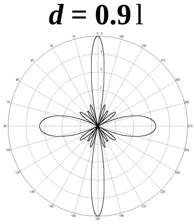
Grating lobes are secondary lobes in the radiation pattern having the same magnitude as the main lobe. When the spacing between array elements becomes too large, grating lobes appear. Grating lobes are a serious detriment to the directivity of an antenna system. To prevent the occurance of grating lobes, the element spacing should be less than one wavelength.

Below are radiation patterns of a 5-element dipole array with spacings d=0.8λ, d=0.9λ, and d=λ. The main lobes are the thin vertical lobes and the grating lobes are the large horizontally opposed lobes.

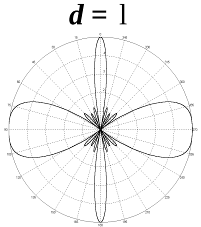
d = 0.8λ



d = 0.9λ



d = λ



VISIBLE REGION

The **Array Factor** is an unbounded function with a period of 2π. However, only a finite region of the function plays a part in the radiation pattern of the array. This is called the visible region and is defined by:

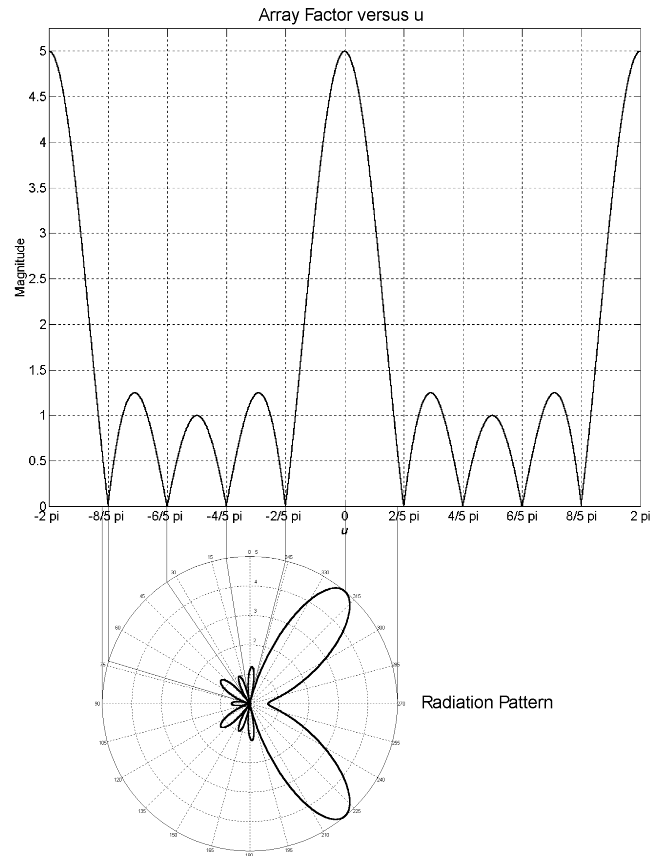
$$\boxed{-k_0 d + \psi \;\le\; u \;\le\; +k_0 d + \psi}$$

ψ = phase angle between adjacent [radians]
k_0 = phase constant in free space [m^{-1}]
d = the distance between adjacent elements [m]

GRAPHICAL METHOD

The **Radiation Pattern** can be determined using the information covered in the preceeding three sections. A polar plot is created below the plot of the **array factor versus *u***. The diameter of the polar plot of the **array factor versus** ϕ is fitted into the visible range. Lines extending downward from the nodes to the perimeter of the polar plot and then to its center determine where the lobes are drawn. For the example below:

N = 5, i.e. a 5-element array
d = λ/2, elements are spaced λ/2 apart
ϕ_0 = 40°, the steering angle
ψ = -2.019 or -0.643π, the phase angle between elements
-1.64π to 0.36π is the visible range



The Matlab code for creating these two plots is in GraphicalMethod.m and requires a second file polar3.m.

FRIIS FREE SPACE TRANSMISSION FORMULA

The Friis transmission formula is used in measuring the gain of antennas and in determining the proper aperture size for an application. The formula expresses the relationship between received and transmitted power, also called the *power transfer ratio*:

$$\boxed{\frac{\langle P_R \rangle}{\langle P_T \rangle} = \left(\frac{\lambda}{4\pi r} \right)^2 (\mathrm{DIR})_T \, (\mathrm{DIR})_R}$$

To determine the size of the antennas needed, it is necessary to know the minimum amount of power required at the receiving antenna:

$$\langle P_R \rangle_{\text{minimum}} = \langle P_N \rangle (\mathrm{S/N})_{\text{minimum}}$$

where $\langle P_N \rangle = kT_s B$

T_s = equivalent system noise "temperature" [K]
k = Boltzmann constant 1.380658×10^{-23} [J/K]
P_T = transmitted power [W]
P_R = power received [W]
r = distance between antennas [m]
$(\mathrm{DIR})_T$ = directivity of the transmitting antenna [no units]
$(\mathrm{DIR})_R$ = directivity of the receiving antenna [no units]

RADAR

Maximum detectable range:

$$\boxed{r_{\max} = \left[\frac{(\mathrm{DIR})^2 \lambda^2 \sigma \langle P_T \rangle \tau}{(4\pi)^3 (\mathrm{S/N})_{\min} (k\,T_s)} \right]^{\frac{1}{4}}} \quad [\mathrm{m}]$$

Radar cross-section:

$$\sigma = \frac{\langle \mathscr{S} \rangle_{\text{back}}}{\langle \mathscr{S} \rangle_{\text{incident}}} (4\pi r^2) = \frac{|E^s|^2}{|E^i|^2} (4\pi r^2) \quad [\mathrm{m}^2]$$

Ambient noise: $\langle P_N \rangle = kT_S B$ [W]

Incident power at the target:

$$\langle \mathscr{S} \rangle_{\text{incident}} = \frac{\langle P_T \rangle}{4\pi r^2} (\mathrm{DIR})_T \quad [\mathrm{W/m^2}]$$

Receiver bandwidth: $B = \frac{1}{\tau}$ [Hz]

T_s = equivalent system noise "temperature" [K]
k = Boltzmann constant 1.380658×10^{-23} [J/K]
τ = duration of transmitted pulse [s]
P_T = transmitted power [W]

TL TRANSMISSION LOSS [dB]

The transmission loss between a transmitting and receiving antenna depends on the antenna gains, the distance and the frequency:

$$\boxed{-TL = (\mathrm{DIR})_T + (\mathrm{DIR})_R - 20\log\frac{4\pi r}{\lambda}} \quad [\mathrm{dB}]$$

$(\mathrm{DIR})_T$ = directivity of the transmitting antenna [no units]
$(\mathrm{DIR})_R$ = directivity of the receiving antenna [no units]
r = distance between antennas [m]
λ = wavelength [meters]

see also PLANE EARTH TRANSMISSION LOSS p25.

NUMERICAL METHODS

COURANT STABILITY CONDITION

The FDTD numerical method for calculating an electromagnetic field requires that the propagation speed of the calculations be equal to or greater than the propagation speed of the wave:

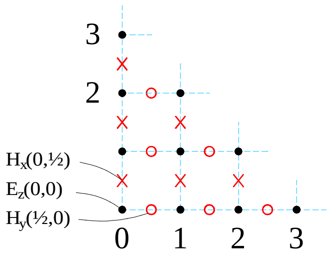
$$\Delta t \le \frac{\delta}{\sqrt{2}} \left(\frac{1}{c} \right)$$

Δt = increment of time between calculations [s]
δ = increment of distance between discrete values, e.g. the distance between (i,j) and $(i+1,j)$ [s]
c = speed of light, (2.998×10^8 m/s in free space)

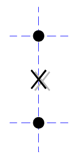
FDTD FINITE DIFFERENCE TIME DOMAIN

In this class, we did a 2D FDTD computer simulation. Refer to the document FDTD-homework10.pdf. The FDTD is the simplest numerical method. It is based on the **Yee Grid**..

The 2-dimensional Yee Grid at right is a coordinate system with electric field components located discretely at integer intercepts in the *xy*-plane and the magnetic components located intermediately as shown.

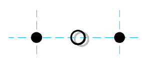


H_x Values of H_x depend on the past value of H_x at the same position and values of E_z above and below.



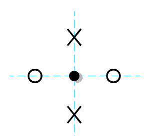
$$H_x^{n+\frac{1}{2}}\left(i, j+\tfrac{1}{2}\right) = H_x^{n-\frac{1}{2}}\left(i, j+\tfrac{1}{2}\right) - \frac{\Delta t}{\delta \mu_0}\left[E_z^n\left(i, j+1\right) - E_z^n\left(i, j\right)\right]$$

H_y Values of H_y depend on the past value of H_y at the same position and values of E_z to the left and right.



$$H_y^{n+\frac{1}{2}}\left(i+\tfrac{1}{2}, j\right) = H_x^{n-\frac{1}{2}}\left(i+\tfrac{1}{2}, j\right) + \frac{\Delta t}{\delta \mu_0}\left[E_z^n\left(i+1, j\right) - E_z^n\left(i, j\right)\right]$$

E_z Values of E_z depend on the past value of E_z at the same position and values of H_x above and below, and values of H_y to the left and right.



$$E_z^{n+1}\left(i, j\right) = E_z^n\left(i, j\right) + \frac{\Delta t}{\delta \varepsilon_0}\left[H_y^{n+\frac{1}{2}}\left(i+\tfrac{1}{2}, j\right) - H_y^{n+\frac{1}{2}}\left(i-\tfrac{1}{2}, j\right) - H_x^{n+\frac{1}{2}}\left(i, j+\tfrac{1}{2}\right) + H_x^{n+\frac{1}{2}}\left(i, j-\tfrac{1}{2}\right)\right] - \frac{\Delta t}{\varepsilon_0} J_z^{n+\frac{1}{2}}\left(i, j\right)$$

Calculations proceed in the order shown, H_x, H_y, then E_z. Special treatment is given to the boundaries.

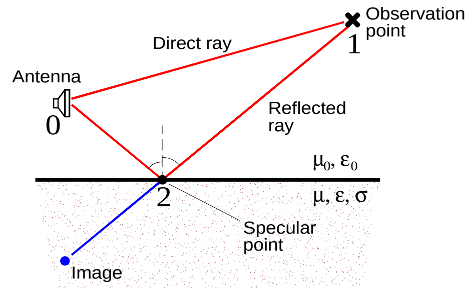
n is used to describe the relative order in time, e.g. $n+½$ occurs after n.

Δt = increment of time between calculations [s]

δ = increment of distance between discrete values, e.g. the distance between (i,j) and $(i+1,j)$ [s]

GEOMETRICAL OPTICS

GEOMETRICAL OPTICS APPROXIMATION



$$\vec{\mathscr{E}}\left(1\right) = \frac{A\left(\theta\right)}{\widehat{01}} e^{-jk_0 \widehat{01}}\hat{y} + \Gamma \frac{A\left(\theta\right)}{\widehat{02}+\widehat{21}} e^{-jk_0\left(\widehat{02}+\widehat{21}\right)}\hat{y}$$

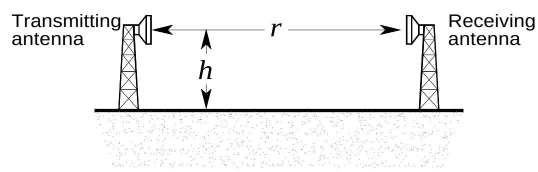
$A(\theta)$ = far field radiation pattern

$\widehat{01}$ = distance from point 0 to point 1

Γ = reflection coefficient, *see REFLECTION COEFFICIENT* p14.

PLANE EARTH TRANSMISSION LOSS

This is an approximation based on geometrical optics, not valid for tall antennas.



$$-TL = 10\log\frac{\langle P_R \rangle}{\langle P_T \rangle} = \left[10\log\left(\mathrm{DIR}\right)_T \left(\mathrm{DIR}\right)_R h^4\right] - 40\log r$$

TL = transmission loss [dB]

(DIR) = directivity, unitless here (not dB)

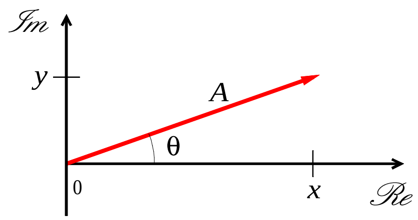
h = antenna height [m]

r = distance between antennas [m]

see also TRANSMISSION LOSS p24.

MATHEMATICS

$x + \mathrm{j}\,y$ COMPLEX NUMBERS



$$x + \mathrm{j}y = A\angle\theta = Ae^{\mathrm{j}\theta} = A\cos\theta + \mathrm{j}A\sin\theta$$

$$\mathcal{Re}\{x + \mathrm{j}y\} = x = A\cos\theta$$

$$\mathcal{Im}\{x + \mathrm{j}y\} = y = A\sin\theta$$

$$\text{Magnitude}\{x + \mathrm{j}y\} = A = \sqrt{x^2 + y^2}$$

$$\text{Phase}\{x + \mathrm{j}y\} = \theta = \tan^{-1}\frac{y}{x}$$

$$j = e^{j\frac{\pi}{2}} = e^{j45°}$$

Expressing a complex number in terms of the natural number e. Note that when using a calculator, the exponent of e must be in radians.

$$a + \mathrm{j}b = a\,e^{+\mathrm{j}b} = a\,e^{+\mathrm{j}(c°)}, \quad \textit{where} \quad b = c/180 \times \pi$$

Taking the square root of a complex number:

$$\sqrt{1 - \mathrm{j}} = \left(\sqrt{2}\; e^{-\mathrm{j}45°}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}} e^{-\mathrm{j}22.5°} = 1.10 - \mathrm{j}0.455$$

With frequency information:

$$\mathcal{Re}\left\{ae^{\mathrm{j}z}e^{\mathrm{j}\omega t}\right\} = a\cos(\omega t + z) \text{ and}$$

$$\mathcal{Re}\left\{-\mathrm{j}ae^{\mathrm{j}z}e^{\mathrm{j}\omega t}\right\} = a\sin(\omega t + z)$$

COMPLEX CONJUGATES

The complex conjugate of a number is simply that number with the sign changed on the imaginary part. This applies to both **rectangular and polar notation**. When conjugates are multiplied, the result is a scalar.

$$(a + jb)(a - jb) = a^2 + b^2$$

$$(A\angle B°)(A\angle -B°) = A^2$$

Other **properties of conjugates**:

$$(ABC + DE + F)^* = (A^*B^*C^* + D^*E^* + F^*)$$

$$(e^{-jB})^* = e^{+jB}$$

TRIG IDENTITIES

$$e^{+\mathrm{j}\theta} + e^{-\mathrm{j}\theta} = 2\cos\theta$$

$$e^{+\mathrm{j}\theta} - e^{-\mathrm{j}\theta} = \mathrm{j}2\sin\theta$$

$$e^{\pm\mathrm{j}\theta} = \cos\theta \pm \mathrm{j}\sin\theta$$

SINC( ) FUNCTION

$$\boxed{\mathrm{sinc}(v) = \frac{\sin(v)}{v}}$$

The sinc function may be involved when finding the ½ -power beamwidth using a field expression. The solution is

$$\frac{\sin(v)}{v} = \frac{1}{\sqrt{2}}, \qquad v = 1.391558 \text{ radians}$$

WORKING WITH LINES, SURFACES, AND VOLUMES

$\rho_l(\mathbf{r}')$ means "the charge density along line l as a function of $\mathbf{r}'$." This might be a value in C/m or it could be a function. Similarly, $\rho_s(\mathbf{r}')$ would be the charge density of a surface and $\rho_v(\mathbf{r}')$ is the charge density of a volume.

For example, a disk of radius a having a uniform charge density of ρ C/m^2, would have a total charge of $\rho\pi a^2$, but to find its influence on points along the central axis we might consider incremental rings of the charged surface as $\rho_s(\mathbf{r}')\,dr' = \rho_s 2\pi r'\,dr'$.

If $d\mathbf{l}'$ refers to an incremental distance along a circular contour C, the expression is $r'd$f, where r' is the radius and df is the incremental angle.

Ñ NABLA, DEL OR GRAD OPERATOR [+ m^-1]

The del operator operates on a scalar to produce a vector. Compare the ∇ operation to taking the time derivative. Where $\partial/\partial t$ means to take the derivative with respect to time and introduces a s^{-1} component to the units of the result, the ∇ operation means to take the derivative with respect to distance (in 3 dimensions) and introduces a m^{-1} component to the units of the result. ∇ terms may be called ***space derivatives*** and an equation which contains the ∇ operator may be called a ***vector differential equation***. In other words ∇A is how fast A changes as you move through space.

in rectangular coordinates: $\nabla A = \hat{x}\frac{\partial A}{\partial x} + \hat{y}\frac{\partial A}{\partial y} + \hat{z}\frac{\partial A}{\partial z}$

in cylindrical coordinates: $\nabla A = \hat{r}\frac{\partial A}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial A}{\partial \phi} + \hat{z}\frac{\partial A}{\partial z}$

in spherical coordinates: $\nabla A = \hat{r}\frac{\partial A}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial A}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial A}{\partial \phi}$

Ñ² THE LAPLACIAN [+ m^-2]

The divergence of a gradient

Laplacian of a **scalar** in **rectangular** coordinates: $\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$

Laplacian of a **vector** in **rectangular** coordinates: $\nabla^2 \vec{A} = \hat{x}\frac{\partial^2 A_x}{\partial x^2} + \hat{y}\frac{\partial^2 A_y}{\partial y^2} + \hat{z}\frac{\partial^2 A_z}{\partial z^2}$

In **spherical** and **cylindrical** coordinates:

$$\nabla^2 \vec{A} \equiv \nabla\left(\nabla \cdot \vec{A}\right) - \nabla \times \nabla \times \vec{A}$$
$$= \text{grad}\left(\text{div}\,\vec{A}\right) - \text{curl}\left(\text{curl}\,A\right)$$

Ñx, div D DIVERGENCE [+ m^-1]

The divergence operation is performed on a vector and produces a scalar. The del operator followed by the dot product operator is read as "the divergence of" and is an operation performed on a vector. In rectangular coordinates, $\nabla\cdot$ means the sum of the partial derivatives of the magnitudes in the x, y, and z directions with respect to the x, y, and z variables. The result is a scalar, and a factor of m^{-1} is contributed to the units of the result.

In the electrostatic context, divergence is the total outward flux per unit volume due to a source charge. For example, in this form of Gauss' law, where $\mathbf{D}$ is a density per unit area, $\nabla\cdot\mathbf{D}$ becomes a density per unit volume.

$$\boxed{\text{div}\,\vec{D} = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho}$$

$\vec{D}$ = electric flux density vector $\vec{D} = \varepsilon\vec{E}$ [C/m²]

ρ = source charge density [C/m³]

in **rectangular** coordinates: $\nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

in **cylindrical** coordinates: $\nabla\cdot\vec{D} = \frac{1}{r}\frac{\partial}{\partial r}\left(rD_r\right) + \frac{1}{r}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$

in **spherical** coordinates:

$$\nabla\cdot\vec{D} = \frac{1}{r^2}\frac{\partial\left(r^2 D_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta D_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

CURL curl $\vec{B} = \nabla \times \vec{B}$ [+ m^{-1}]

The circulation around an enclosed area. The curl operator acts on a vector field to produce another vector field. The curl of vector **B** is
in **rectangular** coordinates:

$$\text{curl } \vec{B} = \nabla \times \vec{B} =$$

$$\hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

This can be written in determinant form and may be easier to remember in this form.

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

in **cylindrical** coordinates:

$$\text{curl } \vec{B} = \nabla \times \vec{B} =$$

$$\hat{r}\left[\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right] + \hat{z}\frac{1}{r}\left[\frac{\partial (rB_\phi)}{\partial r} - \frac{\partial B_r}{\partial \phi}\right]$$

in **spherical** coordinates:

$$\text{curl } \vec{B} = \nabla \times \vec{B} = \hat{r}\ \frac{1}{r\sin\theta}\left[\frac{\partial (B_\phi \sin\theta)}{\partial \theta} - \frac{\partial B_\theta}{\partial \phi}\right] +$$

$$\hat{\theta}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{\partial (rB_\phi)}{\partial r}\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial (rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right]$$

The **divergence of a curl** is always zero:

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = 0$$

DOT PRODUCT [= units^2]

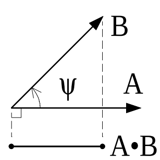
The dot product is a scalar value.

$$\mathbf{A} \bullet \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \bullet (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \bullet \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\psi_{\text{AB}}$$

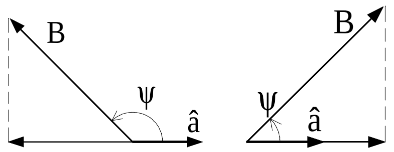
$$\hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = 0, \quad \hat{\mathbf{x}} \bullet \hat{\mathbf{x}} = 1$$

$$\mathbf{B} \bullet \hat{\mathbf{y}} = (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \bullet \hat{\mathbf{y}} = B_y$$



Projection of **B** along **â**:

$$(\mathbf{B} \bullet \hat{\mathbf{a}})\hat{\mathbf{a}}$$



The dot product of 90° vectors is zero.
The dot product is **commutative** and **distributive**:

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A} \qquad \mathbf{A} \bullet (\mathbf{B} + \mathbf{C}) = \mathbf{A} \bullet \mathbf{B} + \mathbf{A} \bullet \mathbf{C}$$

CROSS PRODUCT

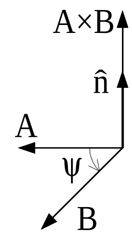
The cross product is an operation performed on two vectors resulting in a third vector perpendicular to the plane in which the first two lie.

$$\mathbf{A} \times \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z)$$

$$= \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x)$$

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}|\mathbf{A}||\mathbf{B}|\sin\psi_{\text{AB}}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to both **A** and **B** (thumb of right-hand rule).



$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$$

$$\mathbf{x} \times \mathbf{y} = \mathbf{z} \qquad \mathbf{y} \times \mathbf{x} = -\mathbf{z} \qquad \mathbf{x} \times \mathbf{x} = 0$$

$$\boldsymbol{\phi} \times \mathbf{z} = \mathbf{r} \qquad \boldsymbol{\phi} \times \mathbf{r} = -\mathbf{z}$$

The cross product is **distributive:**

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

Also, we have:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Cylindrical Coordinates:

$$\hat{r} \times \hat{\phi} = \hat{z} \qquad \hat{\phi} \times \hat{z} = \hat{r} \qquad \hat{z} \times \hat{r} = \hat{\phi}$$

Spherical Coordinates:

$$\hat{r} \times \hat{\theta} = \hat{\phi} \qquad \hat{\theta} \times \hat{\phi} = \hat{r} \qquad \hat{\phi} \times \hat{r} = \hat{\theta}$$

GEOMETRY

SPHERE

Area $A = 4\pi r^2$

Volume $V = \dfrac{4}{3}\pi r^3$

ELLIPSE

Area $A = \pi AB$

Circumference

$$L \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$$

STOKES' THEOREM

S is any unbroken surface (doesn't have to be flat). L is is the closed path (line) around its border. Stokes' theorem says that the line integral of a vector field around the path L is related to the surface integral of the curl on that vector field.

$$\boxed{\oint_L \vec{V} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{V}\right) \cdot d\vec{S}}$$

COORDINATE SYSTEMS

Cartesian or Rectangular Coordinates:

$\mathbf{r}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$ $\quad$ $\hat{x}$ is a unit vector

$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

Spherical Coordinates:

$\mathbf{P}(r, \theta, \phi)$ $\quad$ r is distance from center

θ is angle from vertical

ϕ is the CCW angle from the x-axis

$\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are unit vectores and are functions of position—their orientation depends on where they are located.

Cylindrical Coordinates:

$\mathbf{C}(r, \phi, z)$ $\quad$ r is distance from the vertical (z) axis

ϕ is the CCW angle from the x-axis

z is the vertical distance from origin

COORDINATE TRANSFORMATIONS

Rectangular to Cylindrical:

To obtain: $\mathbf{A}(r, \phi, z) = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$

$A_r = \sqrt{x^2 + y^2}$ $\qquad$ $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\phi = \tan^{-1}\dfrac{y}{x}$ $\qquad$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$z = z$ $\qquad$ $\hat{z} = \hat{z}$

Cylindrical to Rectangular:

To obtain: $\mathbf{r}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$

$x = r\cos\phi$ $\qquad$ $\hat{x} = \hat{r}\cos\phi - \hat{\phi}\cos\phi$

$y = r\sin\phi$ $\qquad$ $\hat{\phi} = \hat{r}\sin\phi + \hat{y}\cos\phi$

$z = z$ $\qquad$ $\hat{z} = \hat{z}$

Rectangular to Spherical:

To obtain: $\mathbf{A}(r, \theta, \phi) = \hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi$

$A_r = \sqrt{x^2 + y^2 + z^2}$

$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$

$$\theta = \frac{z\cos^{-1}}{\sqrt{x^2 + y^2 + z^2}}$$

$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$

$\phi = \tan^{-1}\dfrac{y}{x}$ $\qquad$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

Spherical to Rectangular:

To obtain: $\mathbf{r}(x, y, z) = x\hat{x} + y\hat{y} + z\hat{z}$

$x = r\sin\theta\cos\phi$

$\hat{x} = \hat{r}\sin\theta\cos\phi - \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$

$y = r\sin\theta\sin\phi$

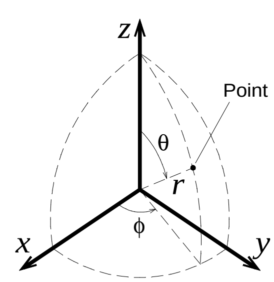
$\hat{y} = \hat{r}\sin\theta\sin\phi$

$\quad + \hat{\theta}\cos\theta\sin\phi$

$\quad + \hat{\phi}\cos\phi$

$z = r\cos\theta$

$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$



VECTOR IDENTITIES

$$\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$$

$$\nabla\cdot\left(\vec{A}+\vec{B}\right) = \nabla\cdot\vec{A}+\nabla\cdot\vec{B}$$

$$\nabla\times\left(\vec{A}+\vec{B}\right) = \nabla\times\vec{A}+\nabla\times\vec{B}$$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot\left(\psi\vec{A}\right) = \vec{A}\cdot\nabla\psi + \psi\nabla\cdot\vec{A}$$

$$\nabla\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}$$

$$\nabla\times\left(\phi\vec{A}\right) = \nabla\phi\times\vec{A} + \phi\nabla\times\vec{A}$$

$$\nabla\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\nabla\cdot\vec{B} - \vec{B}\nabla\cdot\vec{A} + \left(\vec{B}\cdot\nabla\right)\vec{A} - \left(\vec{A}\cdot\nabla\right)\vec{B}$$

$$\nabla\cdot\nabla\phi = \nabla^2\phi$$

$$\nabla\cdot\nabla\times\vec{A} = 0$$

$$\nabla\times\nabla\phi = 0$$

$$\nabla\times\nabla\times\vec{A} = \nabla\left(\nabla\cdot\vec{A}\right) - \nabla^2\vec{A}$$

$$\nabla\left(\vec{A}\cdot\vec{B}\right) =$$
$$\left(\vec{A}\cdot\nabla\right)\vec{B} + \left(\vec{B}\cdot\nabla\right)\vec{A} + \vec{A}\times\left(\nabla\times\vec{B}\right) + \vec{B}\times\left(\nabla\times\vec{A}\right)$$

$$\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right)$$

FOURIER TRANSFORM

The Fourier transform converts a function of time to a function of frequency.

$$\boxed{F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-\mathrm{j}\omega t}\, dt}$$

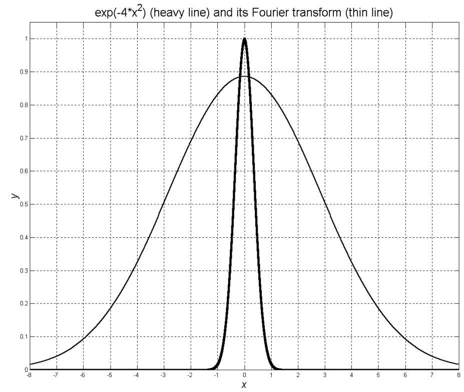
Inverse Fourier transform:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{+\mathrm{j}\omega t}\, d\omega}$$

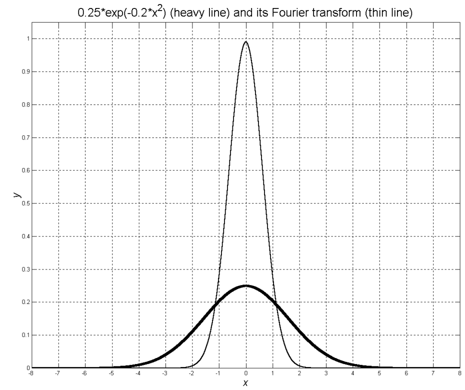
Note that the signs in the exponent for these two functions are shown as they are by convention but they could be reversed as long as one is positive and the other is negative.

FOURIER TRANSFORM EXAMPLES

A gaussian with a sharp peak (heavy line) becomes a gaussian with a "softer" peak after performing the Fourier transform.



A gaussian with a more rounded peak (heavy line below) becomes a gaussian with a sharper peak after performing the Fourier transform.



Taking this to the extreme, the Fourier transform of a constant (horizontal line) becomes a spike and the transform of a spike is a horizontal line. All this is relevant to antennas because in the **farfield expression for a line current** is a *space factor*, which is actually the Fourier transform of the current in the antenna element.

$$\boxed{\vec{\mathscr{E}}^{\,\text{ff}} = \underbrace{\hat{\theta}\,\mathrm{j}\,k_0\eta_0\,\frac{e^{-\mathrm{j}k_0 r}}{4\pi r}\sin\theta}_{\substack{\text{"element pattern"}\\ \text{the field due to an}\\ \text{infinitesimal dipole}}}\ \underbrace{\int_{-h}^{+h} I(z')\, e^{+\mathrm{j}k_0 z'\cos\theta}\, dz'}_{\substack{\text{"space factor"}\\ \text{dependent on current}\\ \text{distribution and length}}}}$$

GLOSSARY

anisotropic materials materials in which the electric polarization vector is not in the same direction as the electric field. The values of ε, μ, and σ are dependent on the field direction. Examples are crystal structures and ionized gases.

anechoic chamber an enclosed space that absorbes radiation so that reflections do not interfere with tests.

dielectric An insulator. When the presence of an applied field displaces electrons within a molecule away from their average positions, the material is said to be polarized. When we consider the polarizations of insulators, we refer to them as *dielectrics*.

empirical A result based on observation or experience rather than theory, e.g. *empirical data*, *empirical formulas*. Capable of being verified or disproved by observation or experiment, e.g. *empirical laws*.

evanescent wave A wave for which $\beta=0$. α will be negative. That is, γ is purely real. The wave has infinite wavelength—there is no oscillation.

incident plane The incident plane is defined by the incident wave vector and a line normal to the boundary surface.

isotropic materials materials in which the electric polarization vector is in the same direction as the electric field. The material responds in the same way for all directions of an electric field vector, i.e. the values of ε, μ, and σ are constant regardless of the field direction.

linear materials materials which respond proportionally to increased field levels. The value of μ is not related to **H** and the value of ε is not related to **E**. Glass is linear, iron is non-linear.

transverse plane perpendicular, e.g. the *x-y* plane is *transverse* to *z*.