INTRODUCTION TO AUTOMATIC CONTROLS

INDEX

LAPLACE TRANSFORMS

We use **Laplace transforms** because we are dealing with **linear dynamic systems** and it is easier than solving differential equations. We don't use **Fourier transforms** because we are dealing with the transient response and because a Fourier transform won't handle a system that "blows up".

LAPLACE TRANSFORM

The Laplace transform is used to convert a function $f(t)$ in the **time domain** to a function $F(s)$ in the ***s* domain**, where s is a complex number:

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$f(t)$ is 0 for $t<0$. $f(t)$ can "blow up" or be piecewise. We are free to *pick* the value of s to make the integral converge; however, once the calculation is made you can use the result everywhere. For example if $f(t) = e^{10t}$, then s must be 10 or greater to do the integration. But the result is $F(s) = 1/(s-10)$, in which s can be less than 10.

Misc: $s = \sigma + j\omega$, $\left| e^{jx} \right| = 1$

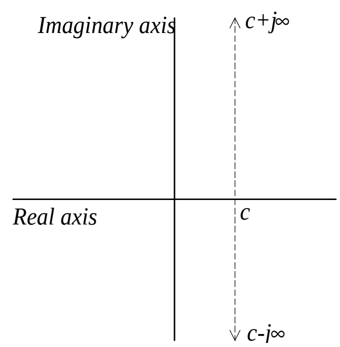
INVERSE LAPLACE TRANSFORM

The inverse Laplace transform is used to convert a function $F(s)$ in the ***s* domain** to a function $f(t)$ in the **time domain**, where s is a complex number:

$$f(t) = \frac{1}{2\pi j} \int_{C-j\infty}^{C+j\infty} F(s) e^{st} ds$$

In the conceptual view, c is a real number defining a line in the s-plane as shown at right. All poles of $F(s)$ must lie to the left of this line.

Poles are always symmetric about the real axis.



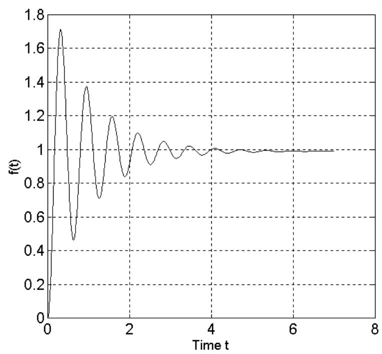
INVERSE LAPLACE TRANSFORM

Second Order Conjugate Pair Example

$$F(s)=\frac{100}{s(s+1+j10)(s+1-j10)}$$

$$f(t)=\frac{10}{101}\left[10-10e^{-t}\cos 10t-e^{-t}\sin 10t\right]$$

A second order conjugate pole pair in the left-hand side of the s-plane results in a damped sinusoid in the time domain.



SYSTEM STABILITY

Stable: A system is stable is there are no roots in the right-hand plane and no repeated roots on the jω axis.

Unstable: A system is unstable if there are any roots in the right-hand plane or repeated roots on the jω axis.

Asymptotically stable: A system is asymptotically (very) stable if all roots are in the left-hand plane.

SOLUTION USING RESIDUES

$$f(t)=\frac{1}{2\pi j}\int_{C-j\infty}^{C+j\infty}F(s)e^{st}ds=\sum \text{residues of } F(s)$$

The inverse Laplace transform can be found by taking the sum of the *residues* of $F(s)$. The function $F(s)$ has a *residue* at each pole of the function. This method requires that the function $F(s)$ have more poles than zeros:

Example:

$$F(s)=\frac{10(s+5)}{s(s-2)}$$

For example, this function has a zero at -5 and poles at 0 and 2. **Zeros** are values for s that cause the numerator to be zero; **poles** are values for s that cause the denominator to be zero.

The residue of $F(s)$ at a simple pole is found by taking the limit as follows:

$$\text{residue}=\lim_{s\to \text{pole}}\left[(s-\text{pole})F(s)e^{st}\right]$$

So for pole=0 in the example above, we have:

$$\lim_{s\to 0}\left[\cancel{(s-0)}\frac{10(s+5)}{\cancel{s}(s-2)}e^{st}\right]=\frac{10(0+5)}{(0-2)}e^{0t}=\frac{50}{-2}$$

and for pole=2 we have:

$$\lim_{s\to 2}\left[\cancel{(s-2)}\frac{10(s+5)}{s\cancel{(s-2)}}e^{st}\right]=\frac{10(2+5)}{2}e^{2t}=\frac{70}{2}e^{2t}$$

So we solve the inverse Laplace transform by

$$f(t)=\sum \text{residues of } F(s)$$

$$f(t)=\left(\frac{50}{-2}+\frac{70}{2}e^{2t}\right)=\left(35e^{2t}-25\right)$$

RESIDUES: REPEATED ROOTS

When there is a repeated root, the procedure for solution using residues changes.

$$\text{residue} = \lim_{s \to \text{pole}} \frac{1}{(n-1)!} \frac{d^{(n-1)}}{ds^{(n-1)}} \left[(s - \text{pole})^n F(s) e^{st} \right]$$

Example: $F(s) = \frac{10(s+5)}{(s-2)^3}$

For example, this function has a zero at -5 and 3 poles at s=2.

$$\text{residue} = \lim_{s \to \text{pole}} \frac{1}{(3-1)!} \frac{d^{(3-1)}}{ds^{(3-1)}} \left[(s-2)^3 \frac{10(s+5)}{(s-2)^3} e^{st} \right]$$

$$= \lim_{s \to \text{pole}} \frac{1}{2} \frac{d^2}{ds^2} \left[(10s+50) e^{st} \right]$$

$$= \lim_{s \to \text{pole}} \frac{1}{2} \frac{d^2}{ds^2} \left[10s\, e^{st} + 50 e^{st} \right]$$

$$= \lim_{s \to \text{pole}} \frac{1}{2} \frac{d}{ds} \left[10st\, e^{st} + 10 e^{st} + 50t\, e^{st} \right]$$

$$= \lim_{s \to \text{pole}} \frac{1}{2} \left[10st^2\, e^{st} + 10t\, e^{st} + 10t\, e^{st} + 50t^2\, e^{st} \right]$$

$$= \frac{1}{2} \left[20t^2\, e^{2t} + 10t\, e^{2t} + 10t\, e^{2t} + 50t^2\, e^{2t} \right]$$

$$= \frac{1}{2} e^{2t} \left[70t^2 + 20t \right] = \left(35t^2 + 10t \right) e^{2t}$$

So we solve the inverse Laplace transform by

$$f(t) = \sum \text{residues of } F(s)$$

and in this case there is only one residue so

$$f(t) = \left(35t^2 + 10t \right) e^{2t}$$

SOLUTION USING DIVISION

This method must be used when the number of zeros is equal or greater than the number of poles.

Example:

$$F(s) = \frac{25(s+3)^2}{s+5}$$

For example, this function has two zeros at -3 and a pole at -5. We carry out the multiplication in the numerator and then divide by the denominator:

$$f(s) = \frac{25s^2 + 150s + 225}{s+5} = 25s + 25 + \frac{150}{s+5}$$

The problem is now divided into three parts:

$F_1(s) = 25s$, $F_2(s) = 25$, and $F_3(s) = \frac{150}{s+5}$

Parts 1 and 2 are done by inspection and part 3 is by residues as before:

$f_1(t) = 25 \frac{d}{dt} \delta(t)$, $f_2(t) = 25\delta(t)$, $f_3(t) = 150e^{-5t}$

This gives the result: $f(t) = 25 \frac{d}{dt} \delta(t) + 25\, \delta(t) + 150e^{-5t}$

note: $\delta(t)$ is the **impulse function**, which is a single input pulse having a large amplitude, short duration, and a plotted area of one.

FINDING THE DIFFERENTIAL EQUATION THAT DESCRIBES A TRANSFER FUNCTION

Example: Given the transfer function:

$$G(s) = \frac{10(s+5)}{s^2(s+1)}$$

Perform the multiplication and, assuming all initial conditions are zero, write:

$$\frac{Y(s)}{R(s)} = \frac{10s+50}{s^3+s^2}$$

Then cross-multiply:

$$s^3 Y(s) + s^2 Y(s) = 10sR(s) + 50R(s)$$

Take the inverse Laplace transform to get:

$$\frac{d^3 y}{dt^3} + \frac{d^2 y}{dt^2} = 10 \frac{dr}{dt} + 50R(t)$$

This differential equation describes the original transfer function above.

What if all initial conditions are not zero?

Example: Given these initial conditions to the transfer function above:

$$y(0)=a$$
$$\frac{d}{dt}y(0)=b$$
$$\frac{d^2}{dt^2}y(0)=c$$

Working backwards in the previous example, take the Laplace transform of each term of the result, incorporating the new initial conditions:

$$\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\}=s^3Y(s)-s^2y(0)-s\frac{d}{dt}y(0)-\frac{d^2}{dt^2}y(0)$$
$$=s^3Y(s)-as^2-bs-c$$
$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\}=s^2Y(s)-as-b$$
$$\mathscr{L}\left\{10\frac{dr}{dt}\right\}=10sR(s)-10a$$
$$\mathscr{L}\left\{50r(t)\right\}=50R(s)$$

So the Laplace transform is:

$$s^3Y(s)-as^2-bs-c+s^2Y(s)-as-b=10sR(s)-10a+50R(s)$$

Grouping terms we get:

$$\left(s^3+s^2\right)Y(s)=10(s+5)R(s)+as^2+as+bs+10a+b+c$$

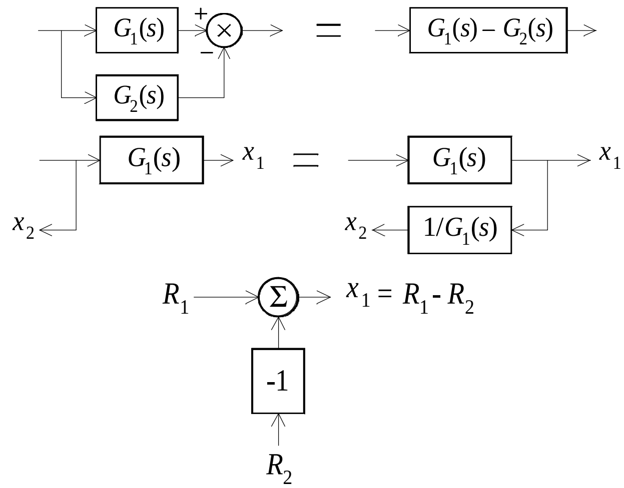
And dividing by (s^3+s^2) gives us the result:

$$Y(s)=\frac{10(s+5)R(s)}{s^2(s+1)}+\frac{a\left(s^2+s+10\right)+b(s+1)+c}{s^2(s+1)}$$

Notice that the first term of the result comes from the original transfer function and the second term is due to the initial conditions.

BLOCK DIAGRAMS

Block diagrams are used to represent transfer function operations of a system. Some basic operations are as follows:



MASON'S GAIN RULE

Mason's gain rule is a method of finding the transfer function of a block diagram. For an example of using Mason's rule, see MasonsRule.pdf.

$$M=\frac{\sum_j M_j\Delta_j}{\Delta}$$

M = transfer function or gain of the system

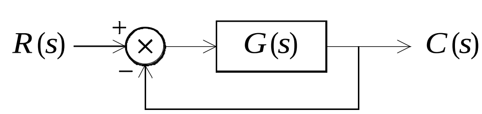
M_j = gain of one forward path

j = an integer representing a forward paths in the system

Δ_j = 1 – the loops remaining after removing path j. If none remain, then Δ_j = 1.

Δ = 1 - Σ loop gains + Σ nontouching loop gains taken two at a time - Σ nontouching loop gains taken three at a time + Σ nontouching loop gains taken four at a time - . . .

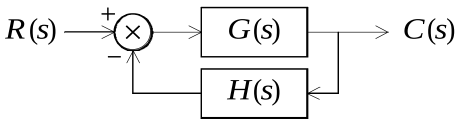
UNITY FEEDBACK SYSTEM



The transfer function for this system is

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

CLOSED LOOP SYSTEM



The transfer function for this system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The transfer function for the **open loop system** (the output is taken to be after $H(s)$) is

$$F(s) = 1+G(s)H(s)$$

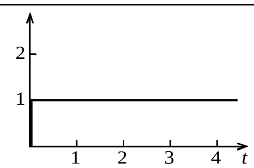
Poles of the closed loop system are zeros of the open loop system. The closed loop system is unstable if $F(s)$ has zeros in the right-hand plane.

$r(t), R(s)$ BASIC TYPES OF INPUTS

Unit step input

$$r(t) = 1,\ t > 0$$

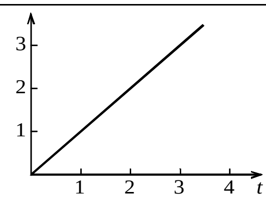
$$R(s) = \frac{1}{s}$$



Unit ramp input

$$r(t) = t,\ t > 0$$

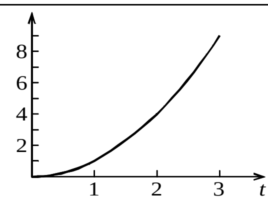
$$R(s) = \frac{1}{s^2}$$



Unit ramp^2 input

$$r(t) = t^2,\ t > 0$$

$$R(s) = \frac{2}{s^3}$$



BASIC TYPES OF SYSTEMS

Type 0 system
- no poles at the origin
- tracks a step input with finite error
- does not track a ramp input
- does not track a square ramp input

Type 1 system
- has one pole at the origin
- tracks a step input with zero error
- tracks a ramp input with finite error
- does not track a square ramp input

Type 2 system
- has two poles at the origin
- tracks a step input with zero error
- tracks a ramp input with zero error
- tracks a square ramp input with finite error

STATE VECTOR MODEL

The state vector model is another method of modeling systems. It is done in the time domain and contains a 1st order differential equation. The solution is a vector.

State Model: $\boxed{\dot{X}(t) = AX(t) + bu(t)}$

for example where A is a 2×2 matrix we would have:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

and this translates to:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t)$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t)$$

The number of elements in the vectors (2 in this case) corresponds to the order of the polynomial in the denominator of the transfer function.

$X(t)$ = state vector, consisting of the output signal and its derivatives
$\dot{X}(t)$ = first derivative of the state vector
A = a square matrix
b = a vector
$u(t)$ = system input signal

Output Equation: $\boxed{c(t) = DX(t)}$

$c(t)$ = system output signal
D = a row vector that always has 1 as the first element and zeros for the remaining elements

We **pick a solution**: $x_1(t) = c(t)$, $x_2(t) = \dot{c}(t)$

The solution is not unique, but it is what we use for this type of problem. For larger than a 2nd order polynomial we would continue with $x_3(t) = \ddot{c}(t)$ etc.

FINDING THE TRANSFER FUNCTION FROM A STATE MODEL

Given the state vector model, the transfer function may be found using the formula:

$$\boxed{C(s) = D[sI - A]^{-1} bU(s)}$$

where I is the identity matrix.

For example, given $\dot{x} = Ax + bu$, $c = Dx$,

$$A = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

we have:

$$C(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(s)$$

$$\frac{C(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{C(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\operatorname{adj} \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix}}{\begin{vmatrix} s+5 & 6 \\ -1 & s \end{vmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For more about finding the **adjoint** of a matrix, see the file Matrices.pdf.

$$\frac{C(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix}}{s(s+5)+6} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{C(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s}{s(s+5)+6} & \frac{-6}{s(s+5)+6} \\ \frac{1}{s(s+5)+6} & \frac{s+5}{s(s+5)+6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

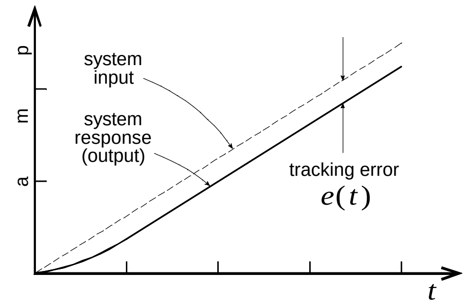
$$\frac{C(s)}{U(s)} = \begin{bmatrix} \frac{s}{s^2+5s+6} & \frac{-6}{s^2+5s+6} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the transfer function is $\dfrac{C(s)}{U(s)} = \dfrac{s-6}{(s+2)(s+3)}$

$e(t)$ TRACKING ERROR

The tracking error is the difference between the input and output of a system.

$$\boxed{e(t) = r(t) - c(t)}$$



$E(s)$ TRACKING ERROR, LAPLACE TRANSFORM

$$\boxed{E(s) = R(s) - C(s)}$$

The Laplace transform of the tracking error of a system.

For the system (no feedback)	The Laplace transform of the tracking error is
$C(s) = G(s)R(s)$	$E(s) = [1 - G(s)]R(s)$
For the system (unity feedback)	**The Laplace transform of the tracking error is**
$C(s) = \dfrac{G(s)}{1+G(s)}R(s)$	$E(s) = \dfrac{1}{1+G(s)}R(s)$

e_{ss} STEADY STATE TRACKING ERROR

The tracking error of a system as $t \to \infty$. The steady state tracking error can be computed from $E(s)$, the LaPlace transform of the tracking error. Note that as $t \to \infty$ in the time domain, $s \to 0$ in the frequency domain.

$$\boxed{e_{ss} = \lim_{s \to 0} sE(s)}$$

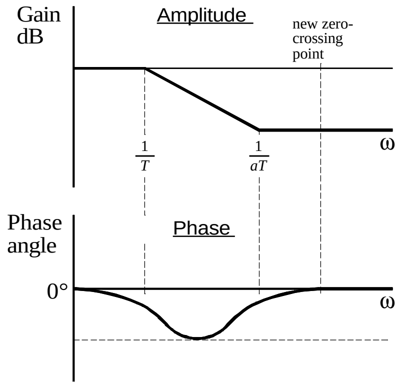
so, for a unity feedback system,

$$e_{ss} = \lim_{s \to 0} s \frac{1}{1+G(s)} R(s)$$

PHASE LAG COMPENSATION

Phase lag compensation reduces the high frequency gain to zero at the location of the desired phase margin.

The **phase lag compensator** shifts the zero crossing downward to the location of the desired phase margin by adding a pole and zero below this point. A negative phase shift occurs, but not at the zero-crossing point.



1) Find the value of K that satisfies the value specified for the steady-state tracking error e_{ss}.

$$e_{ss}(\text{ramp}) = \frac{1}{\lim_{s\to 0}\left[KsG(s)\right]}$$

2) Draw the bode plot of $KG(s)$ and find the frequency at which the desired phase margin occurs. This will be the compensated zero-crossing point ω_0. Determine the amount of **dB gain shift** required to adjust the plot to cross zero at this point (a downward shift is negative).

3) Find the value of a using the dB gain shift found above.

$$20\log a = \text{dB gain shift}$$

4) Now find T.

$$\frac{10}{aT} = \omega_0$$

5) The compensating factor for the system transfer function is:

$$\boxed{G_{\text{lag}}(s) = \frac{1+(aT)s}{1+(T)s}}$$

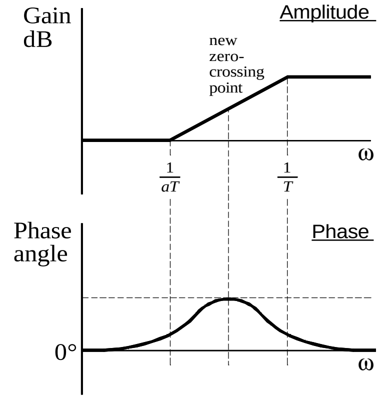
6) And the new transfer function is

$$\boxed{G_{\text{lag}}(s)KG(s)}$$

PHASE LEAD COMPENSATION

Phase lead compensation shifts the zero-crossing point and reduces the phase angle at that point by adding a new pole and zero to the transfer function.

The **phase lead compensator** shifts the zero crossing slightly upward to a point midway between the added pole and zero. The phase plot is bowed upward, with the maximum effect occurring at the new zero-crossing frequency ω_{max}.



1) Find the value of K that satisfies the value specified for the steady-state tracking error e_{ss}.

$$e_{ss}(\text{ramp}) = \frac{1}{\lim_{s\to 0}\left[KsG(s)\right]}$$

2) Draw the bode plot of $KG(s)$ and find the uncompensated phase margin.

3) Find the value of a using the specified phase margin plus a 5° fudge factor and the uncompensated phase margin.

$$\sin\phi_{\text{max}} = \sin\left(\text{PM}_{\text{comp.}} + 5° - \text{PM}_{\text{uncomp.}}\right) = \frac{a-1}{a+1}$$

4) Using a, find the uncompensated gain at the frequency which will become the new zero-crossing point. Note that in this expression a factor of 10 is used instead of 20 because this gain is located midway up the 20 dB/decade slope as shown above.

$$\text{Gain} = -10\log a$$

Find the new zero-crossing point ω_{max} by locating the frequency on the uncompensated bode plot that has the above gain. This will also be the point at which the compensator produces maximum phase shift.

5) Now find T.

$$\omega_{\text{max}} = \frac{1}{T\sqrt{a}}$$

6) The compensating factor for the system transfer function is:

$$\boxed{G_{\text{lead}}(s) = \frac{1+(aT)s}{1+(T)s}}$$

7) And the new transfer function is

$$\boxed{G_{\text{lead}}(s)KG(s)}$$

PID CONTROLLERS

PID stands for *proportional integral derivative*:

$$\overbrace{k_p e(t)}^{\text{proportional}} + \overbrace{K_I \int_0^t e(t)\,dt}^{\text{integral}} + \overbrace{K_d \frac{de}{dt}}^{\text{derivative}}$$

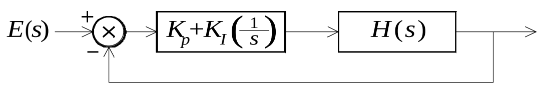
or $K_p + \frac{K_I}{s} + K_d s$

We won't cover this controller, but we will cover the P-D and the P-I controllers.

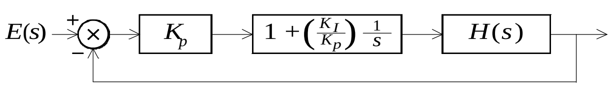
P-I CONTROLLERS

The P-I Controller solution may be obtained using the P-D solution technique.

"P-I" Controller



"P-I" Controller, redrawn



1) Given the transfer function $H(s)$, find the values of K_p and K_d that would achieve P-D compensation for the transfer function $H(s)/s$. These will be the values for K_p and K_I respectively in the P-I controller.

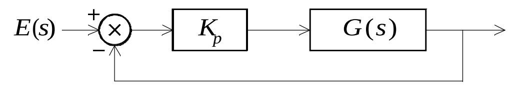
2) The compensated transfer function is

$$K_p\left[1+\left(\frac{K_I}{K_p}\right)\frac{1}{s}\right]H(s)$$

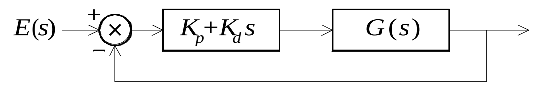
P-D CONTROLLERS

The P-D controller adds a zero at $-(K_p/K_d)$. If less than 45° of phase shift is required then the gain will not change.

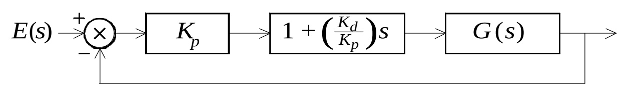
"P" Controller



"P-D" Controller



"P-D" Controller, redrawn



1) Find the value of K_p that satisfies the value specified for the steady-state tracking error e_{ss}.

$$e_{ss}(\text{ramp}) = \frac{1}{\lim_{s\to 0}\left[K_p s G(s)\right]}$$

2) Draw the bode plot of $K_p G(s)$ and find the uncompensated phase margin.

3) If we **do not** need to increase the phase margin by more than 45°, then ω_0 will not change. Use ω_0 from the plot and solve for K_d.

$$\tan\left(\text{PM}_{\text{comp.}} + 5° - \text{PM}_{\text{uncomp.}}\right) = \frac{K_d}{K_p}\omega_0$$

If we **do** need to increase the phase margin by more than 45°, then use the following expression to find the uncompensated gain at the new ω_0. Read the new ω_0 from the plot and plug in to the above expression to find K_d.

$$\text{Gain} = -20\log\sqrt{(1)^2 + \left[\tan\left(\text{PM}_{\text{comp.}} + 5° - \text{PM}_{\text{uncomp.}}\right)\right]^2}$$

4) The compensated transfer function is

$$K_p\left[1+\left(\frac{K_d}{K_p}\right)s\right]G(s)$$

GENERAL

TRIG IDENTITIES

Here are some identities we use:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

GLOSSARY

closed loop system compensates for disturbances by measuring the output response and returning that through a feedback path to compare with the input at the summing junction.

open loop system an input or "reference" is applied to a controller that drives a process. There is no feedback compensation.

PID proportional + integral + derivative, or 3-mode controller.

simple means not repeated or duplicated

steady-state response the approximation to the desired or commanded response

transient response the change from one state to another