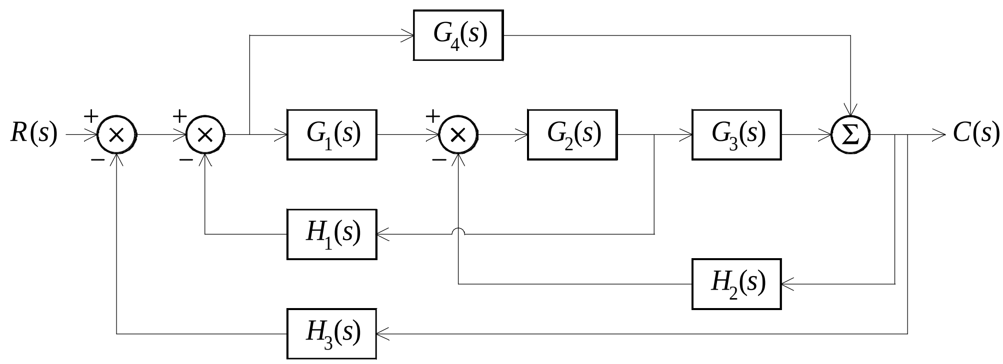
MATRIX SOLUTION

An example of finding the transfer function of a system represented by a block diagram by forming a matrix of equations. This same system has also been solved using Mason's rule; see the file MasonsRule.pdf.

The Problem:

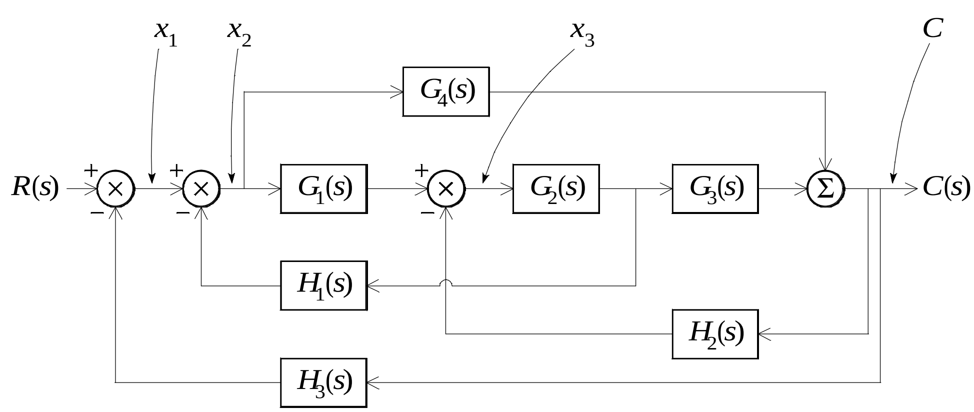
Given the system:



Find an expression for the transfer function $\frac{C(s)}{R(s)}$.

Finding expressions for different part of the system:

Find an expression for the system after each sum or difference and at the output.



Expressions		**Terms isolated from input component**
$x_1 = R - CH_3$	$\rightarrow$	$x_1 + CH_3 = R$
$x_2 = x_1 - x_3 G_2 H_1$	$\rightarrow$	$x_1 - x_2 - x_3 G_2 H_1 = 0$
$x_3 = x_2 G_1 - CH_2$	$\rightarrow$	$x_2 G_1 - x_3 - CH_2 = 0$
$C = x_2 G_4 + x_3 G_2 G_3$	$\rightarrow$	$x_2 G_4 + x_3 G_2 G_3 - C = 0$

Form a matrix equation from the rewritten expressions:

$$\begin{array}{c} \underline{\begin{matrix} C & x_1 & x_2 & x_3 \end{matrix}} \\ \begin{bmatrix} H_3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ -H_2 & 0 & G_1 & -1 \\ -1 & 0 & G_4 & G_2G_3 \end{bmatrix} \end{array} \begin{bmatrix} C \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{c} \underline{R} \\ \begin{bmatrix} R \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

Solve for *C* using Cramer's rule:

To do this, replace the "*C*" column with a copy of the "*R*" column:

$$\begin{array}{c} \underline{\begin{matrix} R & x_1 & x_2 & x_3 \end{matrix}} \\ \begin{bmatrix} R & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ 0 & 0 & G_1 & -1 \\ 0 & 0 & G_4 & G_2G_3 \end{bmatrix} \end{array}$$

And now, by Cramer's rule, *C* will be the determinant of this matrix divided by the determinant of the original 4×4 matrix:

$$C = \frac{\begin{vmatrix} R & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ 0 & 0 & G_1 & -1 \\ 0 & 0 & G_4 & G_2G_3 \end{vmatrix}}{\begin{vmatrix} H_3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ -H_2 & 0 & G_1 & -1 \\ -1 & 0 & G_4 & G_2G_3 \end{vmatrix}}$$

To find the determinant of the numerator, we can most easily work with the first column (since it has three zeros):

$$\det\begin{bmatrix} R & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ 0 & 0 & G_1 & -1 \\ 0 & 0 & G_4 & G_2G_3 \end{bmatrix} = R\times\det\begin{bmatrix} 1 & -1 & -G_2H_1 \\ 0 & G_1 & -1 \\ 0 & G_4 & G_2G_3 \end{bmatrix} = R\left[G_1G_2G_3 + G_4\right]$$

To find the determinant of the denominator, we choose the second column:

$$\det\begin{bmatrix} H_3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -G_2H_1 \\ -H_2 & 0 & G_1 & -1 \\ -1 & 0 & G_4 & G_2G_3 \end{bmatrix} = -1\times\det\begin{bmatrix} 0 & -1 & -G_2H_1 \\ -H_2 & G_1 & -1 \\ -1 & G_4 & G_2G_3 \end{bmatrix} + 1\times\det\begin{bmatrix} H_3 & 0 & 0 \\ 0 & G_1 & -1 \\ -1 & G_4 & G_2G_3 \end{bmatrix}$$

$$= -\left[-1 + G_2H_1H_2G_4 - G_2G_3H_2 - G_1G_2H_1\right] + \left[-H_3G_1G_2G_3 + H_3G_4\right]$$

$$= 1 - G_2H_1H_2G_4 + G_2G_3H_2 + G_1G_2H_1 - H_3G_1G_2G_3 + H_3G_4$$

Then by Cramer's rule:

$$C = \frac{R\left[G_1G_2G_3 + G_4\right]}{1 - G_2H_1H_2G_4 + G_2G_3H_2 + G_1G_2H_1 - H_3G_1G_2G_3 + H_3G_4}$$

And the transfer function is:

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_4}{1 - G_2H_1H_2G_4 + G_2G_3H_2 + G_1G_2H_1 - H_3G_1G_2G_3 + H_3G_4}}$$