

ROUTH HURWITZ ANALYSIS

The Routh Hurwitz analysis tells you how many roots are located in the a) left-hand plane, b) right-hand plane, and c) on the $j\omega$ -axis. The technique is illustrated here with an example. The Routh Hurwitz analysis involves creating an array of values. There are two special cases which may be involved, 1) a row of the array is all zeros (this happens in our example) and 2) a row begins with zero but is not all zeros (this does not occur in our example but is covered on the last page).

THE PROBLEM

Given the system equation

$$s^8 + 2s^7 + 8s^6 + 12s^5 + 20s^4 + 16s^3 + 16s^2 = 0$$

find the number of roots located in the left-hand plane, right-hand plane, and on the $j\omega$ -axis.

FIND THE ZEROS

Since the equation can be factored by s^2 , there are two zeros in the polynomial, i.e. two roots at the origin.

$$s^2(s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16) = 0$$

STARTING THE ROUTH HURWITZ ARRAY

We can drop the s^2 factor and consider only the remaining equation

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

The order of this polynomial is $n=6$, so we will be construction an array having $n+1$ or 7 rows. Taking the coefficients, we construct the first two rows of the array in this order

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & \end{array}$$

THE NEXT ROWS OF THE ROUTH HURWITZ ARRAY

Subsequently, each new row is derived from the two rows immediately above it, which I will call the *working rows*. The first column of the 2×2 matrix in our expressions is always the first column of the two working rows of the Routh Hurwitz array. The remaining column is the column of the working rows just to the right of the position of the unknown. The denominator of the expression is the first number of the lower of the two working rows.

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ b_1 & b_2 & b_3 & \end{array}$$

$$b_1 = \frac{-\begin{vmatrix} 1 & 8 \\ 2 & 12 \end{vmatrix}}{2} = 2$$

$$b_2 = \frac{-\begin{vmatrix} 1 & 20 \\ 2 & 16 \end{vmatrix}}{2} = 12$$

$$b_3 = \frac{-\begin{vmatrix} 1 & 16 \\ 2 & 0 \end{vmatrix}}{2} = 16$$

So now we have

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & \end{array}$$

SPECIAL CASE #1: WHEN A ROW IS ALL ZEROS

For the fourth row

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & 0 \\ c_1 & c_2 & c_3 & \end{array}$$

$$c_1 = \frac{-\begin{vmatrix} 2 & 12 \\ 2 & 12 \end{vmatrix}}{2} = 0$$

$$c_2 = \frac{-\begin{vmatrix} 2 & 16 \\ 2 & 16 \end{vmatrix}}{2} = 0$$

$$c_3 = 0$$

and we have

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & 0 \\ 0 & 0 & 0 & \end{array}$$

We take elements of the last non-zero row and form an **auxiliary equation**:

$$2s^{(n+1-K)-0} + 12s^{(n+1-K)-2} + 16s^{(n+1-K)-4} = 0$$

where n is the order of the polynomial (6 in this case), and K is the row number of the row just above the 0s, from which these values are taken (3 in this case). Note the pattern of subtracting nothing from the first exponent, subtracting 2 from the second, and subtracting 4 from the third, etc. So the auxiliary equation becomes

$$2s^4 + 12s^2 + 16 = 0$$

Now we take the derivative with respect to s

$$\frac{d}{ds}(2s^4 + 12s^2 + 16) = 8s^3 + 24s + 0$$

We replace row 4 with these new coefficients

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & 0 \\ 8 & 24 & 0 & 0 \end{array}$$

THE FIFTH ROW

For the fifth row

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & 0 \\ 8 & 24 & 0 & 0 \\ d_1 & d_2 & d_3 & \end{array}$$

$$d_1 = \frac{-\begin{vmatrix} 2 & 12 \\ 8 & 24 \end{vmatrix}}{8} = 6$$

$$d_2 = \frac{-\begin{vmatrix} 2 & 16 \\ 8 & 0 \end{vmatrix}}{8} = 16$$

$$d_3 = 0$$

So we have

$$\begin{array}{cccc} 1 & 8 & 20 & 16 \\ 2 & 12 & 16 & 0 \\ 2 & 12 & 16 & 0 \\ 8 & 24 & 0 & 0 \\ 6 & 16 & 0 & 0 \end{array}$$

THE SIXTH ROW

For the sixth row

$$\begin{array}{cccc}
 1 & 8 & 20 & 16 \\
 2 & 12 & 16 & 0 \\
 2 & 12 & 16 & 0 \\
 8 & 24 & 0 & 0 \\
 6 & 16 & 0 & 0 \\
 e_1 & e_2 & e_3 &
 \end{array}$$

$$\begin{aligned}
 e_1 &= \frac{-\begin{vmatrix} 8 & 24 \\ 6 & 16 \end{vmatrix}}{6} = 2.667 \\
 e_2 &= 0 \\
 e_3 &= 0
 \end{aligned}$$

So we have

$$\begin{array}{cccc}
 1 & 8 & 20 & 16 \\
 2 & 12 & 16 & 0 \\
 2 & 12 & 16 & 0 \\
 8 & 24 & 0 & 0 \\
 6 & 16 & 0 & 0 \\
 2.667 & 0 & 0 & 0
 \end{array}$$

THE SEVENTH ROW

For the seventh row

$$\begin{array}{cccc}
 1 & 8 & 20 & 16 \\
 2 & 12 & 16 & 0 \\
 2 & 12 & 16 & 0 \\
 8 & 24 & 0 & 0 \\
 6 & 16 & 0 & 0 \\
 2.667 & 0 & 0 & 0 \\
 f_1 & f_2 & &
 \end{array}$$

$$\begin{aligned}
 f_1 &= \frac{-\begin{vmatrix} 6 & 16 \\ 2.667 & 0 \end{vmatrix}}{2.667} = 16 \\
 f_2 &= 0
 \end{aligned}$$

So we have

$$\begin{array}{cccc}
 1 & 8 & 20 & 16 \\
 2 & 12 & 16 & 0 \\
 2 & 12 & 16 & 0 \\
 8 & 24 & 0 & 0 \\
 6 & 16 & 0 & 0 \\
 2.667 & 0 & 0 & 0 \\
 16 & 0 & 0 & 0
 \end{array}$$

We now have the required $n+1$ rows.

THE NUMBER OF ROOTS IN THE RHP

Count the number of sign changes in the first column (0 in this case). This is the number of roots in the right-hand plane.

THE NUMBER OF ROOTS ON THE $j\omega$ -AXIS

Solving the auxiliary equation, we find 4 roots to that equation, so there are 4 roots on the $j\omega$ -axis (so far).

THE NUMBER OF ROOTS IN THE LHP

Since $n=6$ and we have accounted for 4 of these roots, the remaining 2 roots are in the left-hand plane.

THE FINAL RESULT

Returning to the beginning of the problem, we had factored out an s^2 from the original polynomial. These are zeros of the polynomial and are therefore on the $j\omega$ -axis. And since the polynomial is in the denominator of the transfer function, these are additional roots on the $j\omega$ -axis and are added to the other 4 roots we have found. So the final result is 6 roots on the $j\omega$ -axis, 0 roots in the right-hand plane and 2 roots in the left-hand plane.

SPECIAL CASE #2: WHAT IF A ROW BEGINS WITH 0?

This special case did not occur in our example, so it is covered here.

If a row should begin with a zero and the row is not all zeros, we replace this leading zero with an ϵ . This ϵ is considered to be a small positive number and expressions containing ϵ that appear in column 1 of subsequent rows are determined to be positive or negative based on this characteristic.