STATE VECTOR MODEL

An example of finding the state vector model of a system given the transfer function in the s-domain.

The Problem:

Given the transfer function:

$$G(s) = \frac{100}{s^2 + 3s + 1}$$

find **A**, **b**, and **D** of the state vector model.

State Vector Model:

$$\boxed{\dot{X}(t) = AX(t) + bu(t)}$$

$X(t)$ = state vector, consisting of the output signal and its derivatives
$\dot{X}(t)$ = first derivative of the state vector
A = a square matrix
b = a vector
$u(t)$ = system input signal

1) Write the transfer function as a function of output divided by the input:

$$G(s) = \frac{C(s)}{U(s)} = \frac{100}{s^2 + 3s + 1}$$

2) Cross multiply:

$$s^2C(s) + 3sC(s) + C(s) = 100U(s)$$

3) Convert to the time domain (inverse Laplace transform):

$$\ddot{c}(t) + 3\dot{c}(t) + c(t) = 100u(t)$$

4) Pick a solution:

$$\text{Let } x_1(t) = c(t), \quad x_2(t) = \dot{c}(t)$$

The solution is not unique, but we just always use this one. Note that $x_1(t)$ and $x_2(t)$ are elements of the state vector $X(t)$ and that there are two elements in this case because of the 2nd order polynomial in the denominator of the transfer function. If the polynomial was 3rd order, we would include $x_3(t) = \ddot{c}(t)$ and so on.

5) Solve for $\dot{X}(t)$:

We want to get $\dot{X}(t)$ in terms of x and u.

It can be seen from step 4 that $\boxed{\dot{x}_1(t) = x_2(t)}$.

It can also be seen from step 4 that $\dot{x}_2(t) = \ddot{c}(t)$.

Solving the expression in step 3 for $\ddot{c}(t)$ we have $\ddot{c}(t) = 100u(t) - 3\dot{c}(t) - c(t)$.

We can express this in terms of x and u to get $\boxed{\dot{x}_2(t) = 100u(t) - 3x_2(t) - x_1(t)}$

6) Back to the state vector model:

State vector model: $\dot{X}(t) = AX(t) + bu(t)$

State vector model showing matrices: $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$.

Carrying out the operations: $\boxed{\begin{aligned} \dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t) \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t) \end{aligned}}$

7) Plugging in to the state vector model for matrices A and b:

Using the results of steps 5 and 6 we can find the values for the matrices **A** and **b**.

$$\dot{x}_1(t) = x_2(t): \quad \dot{x}_1(t) = \underbrace{a_{11}}_{0} x_1(t) + \underbrace{a_{12}}_{1} x_2(t) + \underbrace{b_1}_{0} u(t)$$

$$\dot{x}_2(t) = 100u(t) - 3x_2(t) - x_1(t): \quad \dot{x}_2(t) = \underbrace{a_{21}}_{-1} x_1(t) + \underbrace{a_{22}}_{-3} x_2(t) + \underbrace{b_2}_{100} u(t)$$

So matrices **A** and **b** are: $\boxed{\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}}$ $\boxed{\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}}$

8) Finding matrix D from the output equation:

The output equation is $c(t) = \mathbf{D}X(t)$.

The output equation in matrix form is $c(t) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Carrying out the multiplication we have $c(t) = d_1 x_1(t) + d_2 x_2(t)$.

We already know from step 4 that $x_1(t) = c(t)$.

So we can determine the matrix values $c(t) = \underbrace{d_1}_{1} x_1(t) + \underbrace{d_2}_{0} x_2(t)$

Therefore $\boxed{\mathbf{D} = \begin{bmatrix} 1 & 0 \end{bmatrix}}$.

As it turns out, matrix **D** is predictable. The first element is always 1 and the remaining elements are zeros. The number of elements is equal to the order of the polynomial in the denominator of the transfer function.