Chapter 15
Functions of Several Variables

Function of Two Variables: 15.1 p841 Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a real number $f(x, y)$, then f is called a **function of *x* and *y***. The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

Function of Three Variables: 15.1 p841 For the function given by $z = f(x, y)$, we call x and y the **independent variables** and z the **dependent variable**.

Neighborhoods in the Plane: 15.2 p853 Using the formula for the distance $\delta > 0$ between two points (x, y) and (x_0, y_0) in the plane, we define the $\delta-\text{neighborhood}$ about (x_0, y_0) to be the **disc** centered at (x_0, y_0) with radius d (that's a "delta").

$$\{(x, y)\text{: } \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$$

When this formula contains the *less than* inequality, the disc is called **open**, and when it contains the *less than or equal to* inequality, the disc is called **closed**. A point (x_0, y_0) in a plane region R is an **interior point** of R if there exists a $d-\text{neighborhood}$ about (x_0, y_0) that lies entirely in R. If every point in R is an interior point, then we call R an **open region**. A point (x_0, y_0) is a **boundary point** of R if every open disc centered at (x_0, y_0) contains points inside R and points outside R. By definition, a region must contain its interior points, but it need not contain its boundary points. If a region contains all its boundary points, then we say that the region is **closed**. A region that contains some but not all its boundary points is neither open nor closed.

Definition of the Limit of a Function of Two Variables: 15.2 p854 Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disc centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for each $e > 0$, there corresponds a $\delta > 0$ such that

$$\left| f(x, y) - L \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Definition of Continuity of a Function of Two Variables: 15.2 p857 A function f of two variables is **continuous at a point** (x_0, y_0) in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Properties of Continuous Functions of Two Variables: 15.2 p857 If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are continuous at (x_0, y_0).

Scalar multiple: kf
Sum and difference: $f \pm g$
Product: fg
Quotient: f/g, if $g(x_0, y_0) \neq 0$

Continuity of a Composite Function: 15.2 p858 If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x, y) = g(h(x, y))$ is continuous at (x_0, y_0). That is,

$$\lim_{(x,y)\to(x_0,y_0)} g(h(x, y)) = g(h(x_0, y_0))$$

Definition of Continuity of a Function of Three Variables: 15.2 p859 A function f of three variables is **continuous at a point** (x_0, y_0, z_0) in an open region R if $f(x, y, z)$ is defined and equal to the limit of $f(x, y, z)$ as (x_0, y_0, z_0) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

Partial Derivatives: 15.3 p863 If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and to y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

In other words, if $z = f(x, y)$, then to find f_x we consider y **constant** and differentiate with respect to x. To find f_y we consider x **constant** and differentiate with respect to y.

Notation for First Partial Derivatives: 15.3 p863 For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial \mathrm{x}} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and $$\frac{\partial}{\partial \mathrm{y}} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

In other words, the values of ¶f / ¶x and ¶f / ¶y at the point (x_0, y_0, z_0) denote the **slope of the surface in the *x* and *y* directions**.

Higher-order Partial Derivatives: 15.3 p867 We denote high-order partial derivatives by the order in which the differentiation occurs. For instance, the function $z = f(x, y)$ has the following second partial derivatives.

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$	1. Differentiate twice with respect to *x*.
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$	2. Differentiate twice with respect to *y*.
$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$	3. Differentiate first with respect to *x* and then with respect to *y*.
$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$	4. Differentiate first with respect to *y* and then with respect to *x*.

Equality of Mixed Partial Derivatives: 15.3 p868 If f is a function of x and y such that f, f_x, f_y, f_{xy}, and f_{yx} are continuous on an open region R, then for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y)$$

Knowledge of **Partial Derivatives** is important for students who will be taking **Differential Equations**.

Increments: For the function $z = f(x, y)$, Δx and Δy are the **increments of *x* and *y***. The **increments of *z*** is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Total Differential: 15.4 p871 If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y)dx + f_y(x, y)dy$$

Differentiability: 15.4 p872 A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be expressed in the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is said to be **differentiable in a region *R*** if it is differentiable at each point of R.

Sufficient condition for differentiability: 15.4 p873 If f is a function of x and y, where f, f_x, and f_y are continuous in an open region R, then f is differentiable on R.

Differentiability implies continuity: 15.4 p876 If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Chain Rule: one independent variable: 15.5 p879 Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Chain Rule: two independent variables: 15.5 p882 Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x / \partial s$, $\partial x / \partial t$, $\partial y / \partial s$, and $\partial y / \partial t$ all exist, then $\partial w / \partial s$ and $\partial w / \partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

Chain Rule: implicit differentiation: 15.5 p884 If the equation $f(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0$$

Directional Derivative: 15.6 p888 Let f be a function of two variables x and y and let $\bar{u} = \cos\theta i + \sin\theta j$ be a unit vector. Then the **directional derivative of *f* in the direction of *u***, denoted by $D_u f$ is

$$D_u f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t}$$

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\bar{u} = \cos\theta i + \sin\theta j$ is

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

An **alternate form** of the directional derivative is:

$$D_u f(x, y) = \nabla f(x, y) \cdot u$$

Gradient of a function of two variables: 15.6 p890 If $z = f(x, y)$, then the **gradient of *f***, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y) i + f_y(x, y) j$$

We read ∇f as "del f". Another notation for the gradient is **grad** $f(x, y)$.

Properties of the **gradient**: 15.6 p892 Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = 0$, then $D_u f(x, y) = 0$ for all u.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_u f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_u f(x, y)$ is $-\|\nabla f(x, y)\|$.

To visualize one of the properties of the **gradient**, imagine a skier coming down a mountainside. If $f(x, y)$ denotes the altitude of the skier, then $-\nabla f(x, y)$ indicates the *compass direction* the skier should take to ski the path of steepest descent. The *gradient* indicates direction in the xy plane and does not itself point up or down the mountainside.

Gradient is normal to level curves: 15.6 p894 If f is differentiable at (x_0, y_0), and $\nabla f(x_0, y_0) \neq 0$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Directional Derivative and Gradient for a function of three variables: 15.6 p895 Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of *f*** in the direction of a unit vector $u = ai + bj + ck$ is given by

$$D_u f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z)$$

The **gradient of *f*** is defined to be

$$\nabla f(x, y, z) = f_x(x, y, z) i + f_y(x, y, z) j + f_z(x, y, z) k$$

Tangent Plane and Normal Line: 15.7 p898 Let F be differentiable at the point $P = (x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that $\nabla F(x_0, y_0, z_0) \neq 0$.

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane** to S at P.
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line** to S at P.

Equation of Tangent Plane: 15.7 p899 If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Symmetric equation of a normal line:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Angle of inclination of a plane: 15.7 p 902

$$\cos q = \frac{\left| n \cdot k \right|}{\left\| n \right\| \left\| k \right\|} = \frac{\left| n \cdot k \right|}{\left\| n \right\|} \quad \text{or}$$

$$\cos q = \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

Gradient is Normal to level surfaces: 15.7 p904 If F is differentiable at (x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq 0$, then $\nabla F(x_0, y_0, z_0)$ is normal to the level surface through (x_0, y_0, z_0).

Extreme Value Theorem: 15.8 p906 Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R where f takes on a minimum value.
2. There is at least one point in R where f takes on a maximum value.

Relative Extrema: 15.8 p906 Let f be a function defined on a region R containing (x_0, y_0).

1. $f(x_0, y_0)$ is a **relative minimum** of f if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in an *open* disc containing (x_0, y_0).
2. $f(x_0, y_0)$ is a **relative maximum** of f if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in an *open* disc containing (x_0, y_0).

Critical Point: 15.8 p906 Let f be defined on an open region R containing (x_0, y_0). We call (x_0, y_0) a **critical point** of f if one of the following is true.

Relative extrema occur only at Critical Points: 15.8 p907 If $f(x_0, y_0)$ is a relative extremum of f on an open region R, then (x_0, y_0) is a critical point of f.

Second-Partials Test for relative extrema: 15.8 p909 Let f have continuous first and second partial derivatives on an open region containing a point (a, b) for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$. To test for relative extrema of f, we define the quantity

$$d = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a **relative minimum**.
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a **relative maximum**.
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point**.
4. The test gives no information if $d = 0$.

Least squares regression line: 15.9 p916 The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{and} \quad b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i\right)$$

Lagrange's Theorem: Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq 0$, then there is a real number λ (lambda) such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Method of Lagrange multipliers: 15.10 p921 Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. The method may be used for functions of multiple variables. To find the minimum or maximum of f, use the following steps.

1. Simultaneously solve the equations $g(x, y) = c$ and $\nabla f(x, y) = l \nabla g(x, y)$ by solving the following system of equations.

   $f_x(x, y) = l g_x(x, y)$ (solve for lambda)

   $f_y(x, y) = l g_y(x, y)$ (solve for lambda)

   Set the two equal to each other and solve for variables.

   $g(x, y) = c$ (substitute in variables)

   Substitute the results into the original equation.

2. Evaluate f at each solution point obtained in the first step and at each endpoint (if any) of the constraint curve. The largest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the smallest value yields the minimum of f subject to the constraint $g(x, y) = c$.