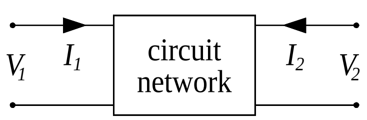
CIRCUIT THEORY EE411

INDEX

TWO-PORT CIRCUITS



$$V_1 = z_{11}I_1 + z_{22}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$
$$V_1 = z_{12}I_2 + z_{11}I_1 ?$$

$$z = \frac{1}{y}$$

$$I_1 = y_{11}V_1 + y_{22}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$
$$I_2 = y_{12}V_2 + y_{11}V_1 ?$$

$$V_1 = h_{11}I_1 + h_{22}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$
$$I_1 = h_{12}I_2 + h_{11}V_1 ?$$

To calculate the z parameters in a resistive network the equations above are manipulated to the following form, where one of the currents is held to zero. Various manipulations are carried out to find values for the V and I quantities using the known values of the resistors.

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

h parameters are used for transistor specifications
y parameters may be easier to find than z parameters and may be added when networks are paralleled.

RMS

rms stands for **root mean square**. To obtain the rms value of a periodic function, first square the function, then take the mean value, and finally the square root.

rms by definition: $$X_{rms} = \sqrt{\frac{1}{T}\int_0^T \left(f(x)\right)^2 dt}$$

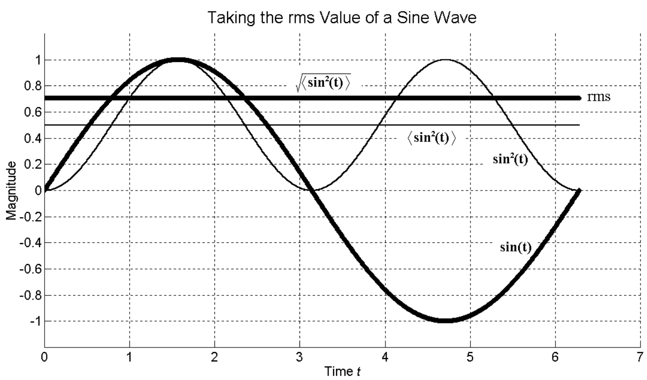
rms value of AC voltage: $$V_{rms} = \frac{V_{\max}}{\sqrt{2}}$$

root mean square

$$\sqrt{\langle p^2 \rangle}$$

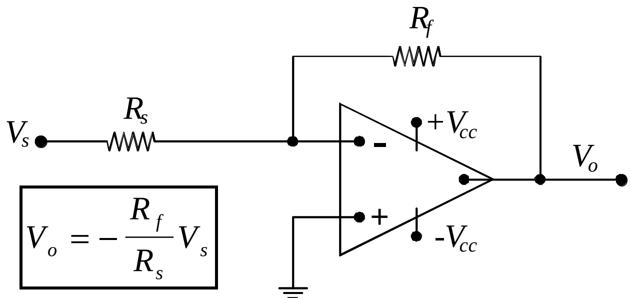
$$f(t)_{\mathrm{rms}} = \sqrt{\left\langle f(t)^2 \right\rangle}$$

The plot below shows a sine wave and its rms value, along with the intermediate steps of squaring the sine function and taking the mean value of the square. Notice that for this type of function, the mean value of the square is ½ the peak value of the square.



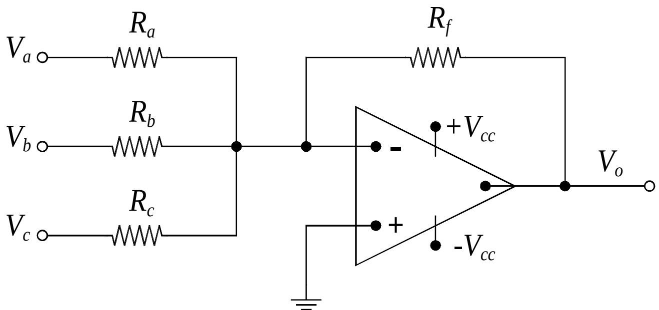
Op Amps

INVERTING AMPLIFIER



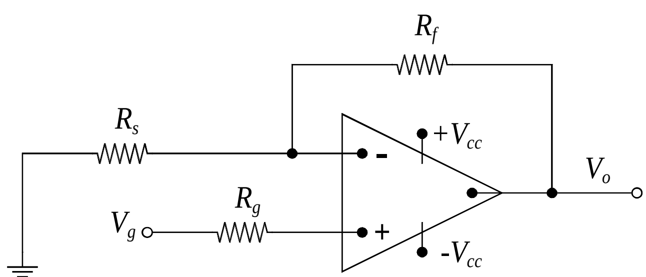
$$V_o = -\frac{R_f}{R_s} V_s$$

INVERTING SUMMING AMPLIFIER



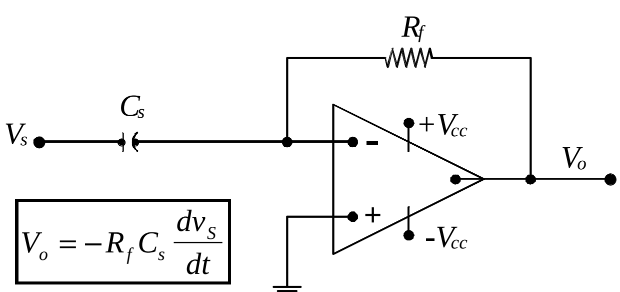
$$V_o = -\left(\frac{R_f}{R_a} V_a + \frac{R_f}{R_b} V_b + \frac{R_f}{R_c} V_c \right)$$

NONINVERTING AMPLIFIER



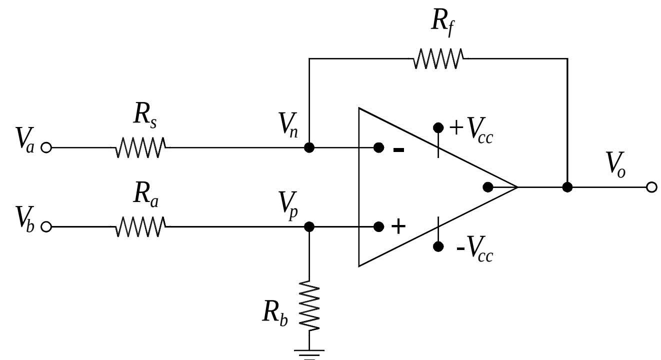
$$V_o = \frac{R_s + R_f}{R_s} V_g$$

DIFFERENTIATING AMPLIFIER



$$V_o = -R_f C_s \frac{dv_S}{dt}$$

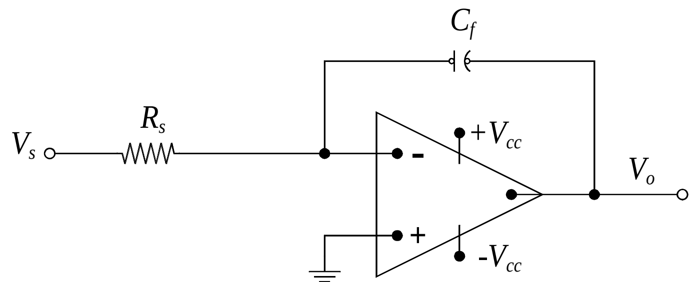
DIFFERENCE AMPLIFIER



$$V_n = V_p = V_b \frac{R_b}{R_a + R_b}$$

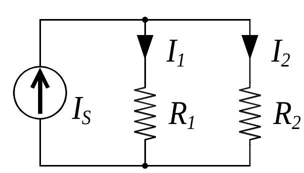
$$\frac{V_n - V_a}{R_s} + \frac{V_n - V_o}{R_f} = 0$$

INTEGRATING AMPLIFIER



$$V_o = -\frac{1}{R_s C_f} \int_{t_o}^{t} V_s \quad d\tau + V_o(t_o)$$

CURRENT DIVISION



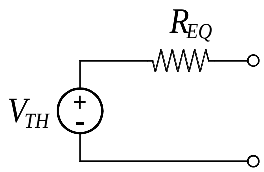
$$I_1 = \frac{R_2}{R_1 + R_2} I_S$$

$$I_2 = \frac{R_1}{R_1 + R_2} I_S$$

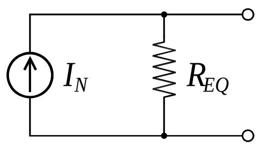
THÈVENIN AND NORTON EQUIVALENTS

A **one-port network** (circuit presenting 2 external terminals) may be represented by either a Thèvenin or Norton equivalent. Note that R_{EQ} has the same value in both the Thèvenin and Norton equivalents.

THÈVENIN EQUIVALENT



NORTON EQUIVALENT

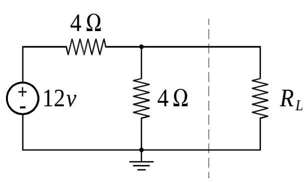


1) Find the Thèvenin voltage (the open-circuit voltage) or the Norton current (the short-circuit current).
2) To find R_{eq}, first "Turn off" the independent sources, i.e. voltage sources go to zero which means they are shorted and current sources also go to zero which means they are opened. Calculate the equivalent resistance of the circuit. This is R_{eq}.
3) If there are no independent sources (dependent sources may be present) then $V_{TH} = I_N = 0$ and the circuit reduces to an equivalent resistance.
4) If there are independent and dependent sources, turn off the independent sources and apply a test source ($V_{TEST} = 1$ or $I_{TEST} = 1$) to the port. Calculate the unknown parameter V_{TEST} or I_{TEST} at the port and find R_{EQ} using

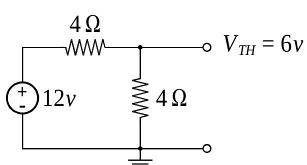
$$V_{TEST} = I_{TEST} R_{EQ}$$

THÈVENIN/NORTON EXAMPLE

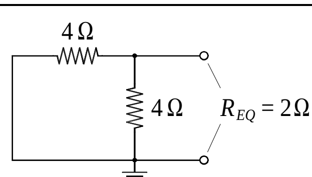
Given this circuit:



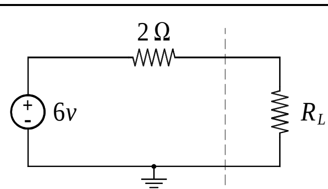
The Thèvenin voltage is the open circuit voltage, i.e. with the load disconnected.



The Thèvenin resistance is the equivalent resistance with the independent voltage source shorted and the load disconnected.

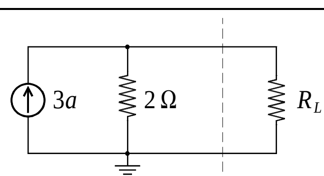


The Thèvenin equivalent circuit can now be written as:

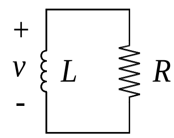


$$V_{TH} = I_N R_{TH}$$

And the Norton equivalent can be written as:



LC CIRCUITS



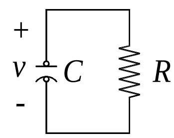
Energy (*joules*): $w = \frac{1}{2} L i^2$

also: $w = \frac{1}{2} L I_o^{\,2} (1 - e^{-2t/\tau})$

Time Constant: $\tau = L / R$

Voltage: $v_L(t) = L \frac{di}{dt}$

Current: $I_L(t) = \frac{1}{L} \int_0^t v \, d\tau + I_o$



Power:
$P = Cv \frac{dv}{dt}$

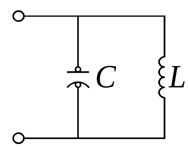
Energy (*joules*): $w = \frac{1}{2} C e^2$

also: $w = \frac{1}{2} C V_o^{\,2} (1 - e^{-2t/\tau})$

Time Constant: $\tau = RC$

Voltage: $V_c(t) = \frac{1}{C} \int_0^t i \, d\tau + V_o$

Current: $i_c(t) = C \frac{dv}{dt}$



LC Tank Circuit

Resonant frequency:

$$f = \frac{1}{2p\sqrt{LC}}$$

Equations Common to L & C Circuits

Current: $i(t) = I_f + (I_o - I_f) e^{-t/\tau}$

Voltage: $v(t) = V_f + (V_o - V_f) e^{-t/\tau}$

Power: $p = I_o^{\,2} R e^{-2t/\tau}$

where I_0 is initial current [A]
I_f is final current [A]
t is time [s]
τ is the time constant; τ = RC for capacitive circuits, τ = R/L for inductive circuits [s]
V_0 is initial voltage [V]
V_f is final voltage [V]
p is power [W]
R is resistance [Ω]

RLC CIRCUITS -- Parallel

Sum of node currents in a Parallel RLC circuit:

$$C\frac{dv}{dt}+\frac{v}{R}+\frac{1}{L}\int_0^t v\,d\tau+I_o=0$$

which differentiates to:

$$C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1}{L}v=0$$

RLC CIRCUITS -- Series

Sum of voltages in a Series RLC circuit:

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int_0^t i\,d\tau+V_o=0$$

which differentiates to:

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=0$$

$$\frac{dx}{dt}\Leftrightarrow j\omega\,\mathrm{X} \qquad \frac{d^2x}{dt^2}\Leftrightarrow (j\omega)^2\,\mathrm{X}$$

RLC CIRCUITS – solving second order equations

a **the Neper frequency (damping coefficient) [rad/s]:**

Parallel circuits: $\alpha=\frac{1}{2RC}$	Series circuits: $\alpha=\frac{R}{2L}$

w **the Resonant frequency [rad/s]:**

$\omega_o=\frac{1}{\sqrt{LC}}$	$\omega_d=\sqrt{\omega_o^{\,2}-\alpha^2}$	used in underdampe d calculations

s_1 **,** s_2 **the roots of the characteristic equation [rad/s]:**

$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^{\,2}}$	$s_2=-\alpha-\sqrt{\alpha^2-\omega_o^{\,2}}$

Overdamped a**²** **>** w**²** (real and distinct roots)

$$X(t)=X_f+A_1'e^{s_1t}+A_2'e^{s_2t}$$

$$X(0)=X_f+A_1'+A_2' \qquad \frac{dx}{dt}(0)=s_1A_1'+s_2A_2'$$

Underdamped a**²** **<** w**²** (complex roots)

$$X(t)=X_f+B_1'e^{-\alpha t}\cos\omega_d t+B_2'e^{-\alpha t}\sin\omega_d t$$

$$X(0)=X_f+B_1' \qquad \frac{dx}{dt}(0)=-\alpha B_1'+w_dB_2'$$

Critically Damped a**²** **=** w**²** (repeated roots)

$$X(t)=X_f+D_1'te^{-\alpha t}+D_2'e^{-\alpha t}$$

$$X(0)=X_f+D_2' \qquad \frac{dx}{dt}(0)=D_1'-\alpha D_2'$$

Some Trig Identities

$$A\cos\omega t+B\sin\omega t=\sqrt{A^2+B^2}\cos\left[\cot+\tan\left(\frac{-B}{A}\right)\right]$$

$c^{\pm j\theta}=\cos\theta\pm j\sin\theta$ Euler identity

$\sin\omega t=\cos(\omega t-90°)$

In an **overdamped** circuit, $\alpha^2>\omega^2$ and the voltage or current approaches its final value without oscillation.

In an **underdamped** circuit, $\alpha^2<\omega^2$ and the voltage or current oscillates about its final value.

In a **critically damped** circuit, $\alpha^2=\omega^2$ and the voltage or current is on the verge of oscillating about its final value.

When an expression is integrated, it may be necessary to add in initial values for the constant of integration even if they have been taken into account within other terms.

Natural response is the behavior of a circuit without external sources of excitation.

Step response is the behavior of a circuit with an external source.

A **node** is a point where two or more circuit elements join.

An **essential node** is a node where three or more circuit elements join.

A **path** is a trace of adjoining basic elements with no elements included more than once.

A **branch** is a path that connects two nodes.

An **essential branch** is a path which connects two essential nodes without passing through an essential node.

A **loop** is a path whose last node is the same as the starting node.

A **mesh** is a loop that does not enclose any other loops.

SINUSOIDAL ANALYSIS

$\pi\times\text{degrees}=180\times\text{radians}$

$\omega=2\pi f\ [\text{rad/s}]=360f\ [\text{deg/s}]$

resonant frequency $\omega_o=\frac{1}{\sqrt{LC}}$

$v(t)=V_m\cos(\omega t+\phi) \qquad i(t)=I_m\cos(\omega t+\phi)$

where V_m and I_m are maximums

equivalent of two parallel impedances $=\frac{\text{product}}{\text{sum}}$

Phasor Transform:	$\mathrm{V}=V_me^{j\phi}=\mathscr{P}\{V_m\cos(\omega t+\phi)\}$ $v(t)=A\cos(\omega t+\phi°)\Leftrightarrow A\angle\phi°$ $\sin\omega t=\cos(\omega t-90°)$
Inverse Phasor Transform	$\mathscr{P}^{-1}\{V_me^{j\phi}\}=\mathscr{R}\{V_me^{j\phi}e^{j\omega t}\}$

A smaller ϕ causes a right shift of the sinusoidal graph.

SINUSOIDAL ANALYSIS

Element:	**Resistor**	**Capacitor**	**Inductor**
Impedance (*Z*):	R (resistance)	$-j/\omega C$	$j\omega L$
Reactance (*X*)	--	$-1/\omega C$	ωL
Admittance (*Y*):	G (conductance)	$j\omega C$	$1/\ j\omega L$
Susceptance:	--	ωC	$-1/\omega L$
Voltage:	$\mathbf{I}R$	$\mathrm{I}/\ j\omega C$ $(I_m\ /\ \omega\ C)\angle(\theta_V\ -90°)$	$j\omega L\mathrm{I}$ $\omega L I_m\ \angle(\theta_V+90°)$
Amperage:	$\mathbf{V}/R$	$j\omega C\mathrm{V}$ $(V_m/\omega C)\angle(\theta_V+90°)$	$\mathrm{V}/\ j\omega L$ $(V_m/\omega L)\angle(\theta_V-90°)$

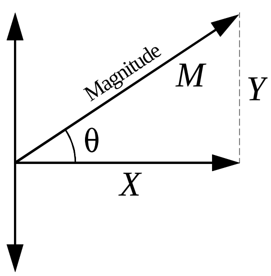
PHASOR and RECTANGULAR NOTATION

The *phasor* is a complex number that carries the amplitude and phase angle information of a sinusoidal function. The Phasor concept is rooted in Euler's identity, which relates the exponential function to the trigonometric function:

$$e^{\pm j\theta}=\cos\theta\pm j\sin\theta$$

The use of phasor notation may be referred to as working in the *phasor domain* or the *frequency domain*. Note that the phasor notation $M\angle\phi$ is equivalent to $Me^{j\phi}$, where ϕ is in radians.

Rectangular Notation: $X\pm jY$ where X represents the horizontal or *real* coordinate and Y the vertical or *imaginary* coordinate. Use this form for addition and subtraction by separately adding and subtracting the real and imaginary components. Be careful with the sign of the **j** term:



$$(A+jB)+(C-jD)=(A+C)+j[B+(-D)]$$

Phasor Notation: $M\angle\phi°$, where M is the magnitude of the phasor and f is the angle CCW from the X axis. Use this form for multiplication and division.

$$(E\angle\theta)(F\angle\phi)=EF\angle(\theta+\phi) \qquad \frac{E\angle\theta}{F\angle\phi}=\frac{E}{F}\angle(\theta-\phi)$$

A negative magnitude may be converted to positive by adding or subtracting 180° from the angle.

To convert from **rectangular to phasor** notation:

Rectangular form: $X\pm jY$

Magnitude: $M=\sqrt{X^2+Y^2}$

Angle ϕ: $\tan\phi=\dfrac{Y}{X}$ (Caution: The *Y* will be negative is the **j** value is being subtracted from the real.)

Note: Due to the way the calculator works, if X is negative, you must **add 180°** after taking the inverse tangent. If the result is greater than 180°, you may optionally subtract 360° to obtain the value closest to the reference angle.

To convert from **phasor to rectangular** (j) notation:

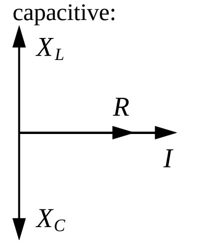
Phasor form: $M\angle\phi°$

X (real) Value: $M\cos\phi$

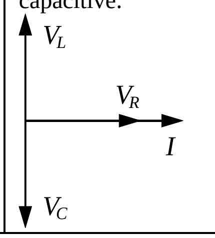
Y (**j** or imaginary) Value: $M\sin\phi$

In conversions, the j value will have the same sign as the θ value for angles having a magnitude < 180°.

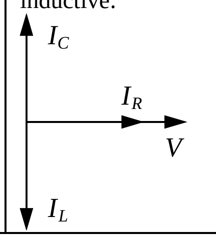
Phasor impedance diagram for series circuits, where +**j** is inductive and -**j** is capacitive:



Phasor voltage diagram for series circuits, where +**j** is inductive and -**j** is capacitive:



Phasor amperage diagram for parallel circuits, where +**j** is capacitive and -**j** is inductive:



POWER

Average Power or real power (*watts*)

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = V_{rms} I_{rms}\cos(\theta_v - \theta_i)$$

Positive P means the load is absorbing **average power,** negative means delivering or generating.

Reactive Power (*VARS*)

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) = V_{rms} I_{rms}\sin(\theta_v - \theta_i)$$

Positive Q means the load is absorbing **magnetizing vars** (inductive), negative means delivering (capacitive).

Complex Power (*VA*)

$$S = P + jQ = V_{rms} I_{rms}(\theta_v - \theta_i)$$

$$= \mathbf{V}_{rms}\mathbf{I}^*_{rms} = \frac{1}{2}\mathbf{V}_{\max}\mathbf{I}^*_{\max} = \frac{\mathbf{V}^2_{rms}}{\mathbf{Z}^*}$$

* means "the complex conjugate of"

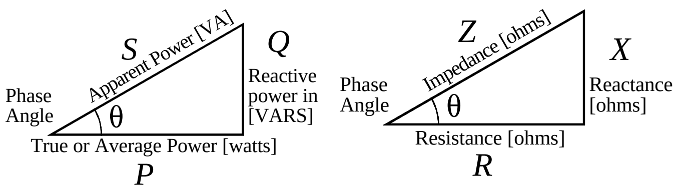
Power Factor (ratio of true power to apparent power)

$pf = \cos(\theta_v - \theta_i)$

Lagging: Inductive, current lags (-j), +Q

Leading: Capacitive, current leads (+j), -Q

Power and Impedance triangles



$\theta = \cos^{-1} pf$ The power triangle is geometrically identical to the impedance triangle.

Maximum Power Transfer

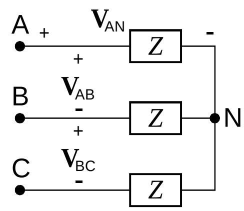
Maximum power transfer occurs when the load impedance is equal to the complex conjugate of the source impedance. Under these conditions, the maximum value of average power absorbed is

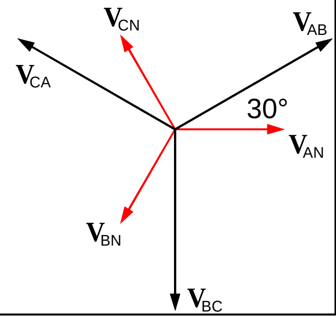
$$P_{\max} = \frac{|V_{TH}|^2}{4R_L}$$

3-PHASE POWER

Phase and line voltage relationships in a Wye Circuit

(positive sequence - clockwise)



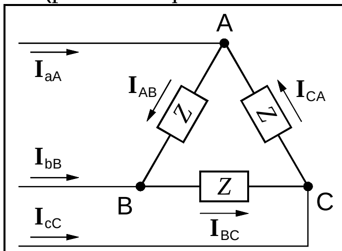


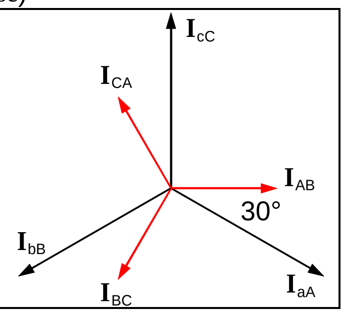
$V_{AB} = V_{AN}\sqrt{3}$ or

$\mathbf{V}_{AB} = \mathbf{V}_{AN}\sqrt{3}\times(1\angle 30°)$

Phase and line current relationships in a Delta Circuit

(positive sequence - clockwise)





$I_{aA} = I_{AB}\sqrt{3}$

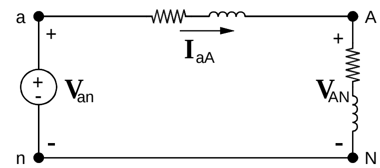
Wye-Delta Transform (for balanced circuits only)

$$Z_Y = \frac{Z_D}{3}$$

$$V_{line-to-line} = V_{an}\sqrt{3}$$

$$I_{aA} = \frac{V_{an}}{Z_f}$$

Single-phase Equivalent Circuit:



Motor Ratings

$$P = \sqrt{3}V_L I_L \cos(\theta_v - \theta_i) = \frac{hp \times 746}{\text{efficiency}}$$ where:

P is the power input in watts

$\cos(\theta_v - \theta_i)$ is the power factor

efficiency is expressed as a decimal value

Power Factor Correction

$$Q = \frac{\text{VARS}}{3} = \frac{(460/\sqrt{3})^2}{x_c = -1/\omega C}$$ where:

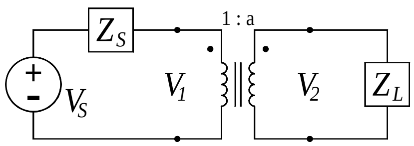
VARS is a negative value for the amount of correction

460 is the line voltage

C is the value of the capacitor in Farads

TRANSFORMERS

Ideal Transformer



$$a = \frac{N_1}{N_2}$$

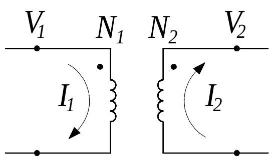
$$V_1 = \frac{V_2}{a}$$

$$Z_{IN} = \frac{V_1}{I_1} = \frac{1}{a^2}\frac{V_2}{I_2} = \frac{1}{a^2} Z_L$$

$$I_1 = aI_2$$

Z_{IN} is the load seen by the source.

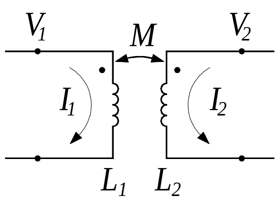
Transformer Turns Ratio



$$\frac{V_1}{N_1} = \frac{V_2}{N_2} \qquad N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

Moving a dot results in a negative sign on one side of each equation.

Magnetically Coupled Coils

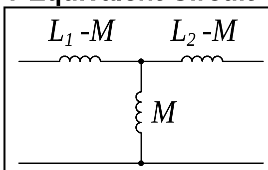


$$V_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$V_2 = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

$M = k\sqrt{L_1 L_2}$ where k is the coefficient of coupling

T Equivalent Circuit



This circuit is equivalent to the magnetically coupled coils above. We are not concerned about the orientation of the dots.

Mesh Current Equations involving mutual inductors

A mesh is a loop that does not enclose other loops in the circuit.

1. Draw current loops emanating from positive voltage sources if present and label I_1, I_2, I_3, etc. for each interior path of the circuit.
2. For each loop form an equation in the form: Voltage or 0 if there is no source in the loop = R_1 × (sum of amperages passing through R_1) + $L_1 \frac{d}{dt}$ × (sum of amperages passing through L_1) + . . .
3. Amperages are positive in the direction of loops regardless of the location of dots on inductors in the loop. However, the sign of an amperage through a mutual inductor is positive iff it enters the mutual inductor at the same end (i.e. dotted or undotted) at which the reference current loop enters the reference inductor.