

EIGENVECTORS

Method for solving systems of first order differential equations using eigenvalues and eigenvectors

9. p288

$$\begin{aligned} x_1' &= 2x_1 - 5x_2 & x_1(0) &= 2, & x_2(0) &= 3 \\ x_2' &= 4x_1 - 2x_2 \end{aligned}$$

The problem can be rewritten $x' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix} x$

Definition: An **eigenvalue** of matrix **A** is a number λ such that $|\mathbf{A} - \lambda \mathbf{I}| = 0$

To find the eigenvalues we rewrite the matrix $\begin{vmatrix} 2-\lambda & -5 \\ 4 & -2-\lambda \end{vmatrix}$

and find its determinant $(2-\lambda)(-2-\lambda) - (-5)(4)$

which we simplify and set equal to zero $\lambda^2 + 16 = 0$

Solving for λ we get $\lambda = \frac{0 \pm \sqrt{0 - 64}}{2}$ yielding complex eigenvalues $\lambda = 0 \pm 4i$

Definition: An **eigenvector** associated with the eigenvalue λ is a nonzero vector \mathbf{v} such that $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ so that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0$

To find the eigenvectors we set up the equation $\begin{bmatrix} 2-(4i) & -5 \\ 4 & -2-(4i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

using the eigenvalue $0 + 4i$

Since in this case we have complex eigenvalues it will be necessary to find only one eigenvector to solve the problem. Otherwise we would need to repeat these steps using the second eigenvalue to find a second eigenvector.

The expression above gives us the two equations	$a(2 - 4i) + (-5)b = 0$	$4a + (-2 - 4i)b = 0$
rewriting we have	$a(2 - 4i) = 5b$	$4a = (2 + 4i)b$
looking at the first equation it appears that if we let $a = 5$ then $b = (2 - 4i)$	$5(2 - 4i) = 5(2 - 4i)$	$4(5) = (2 + 4i)(2 - 4i)$
solving the second equation to verify		$20 = 4 + 16$

therefore the eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 2 - 4i \end{bmatrix}$

since the textbook says that $\mathbf{x}(t) = \mathbf{v}e^{t}$ we can write $\mathbf{x}(t) = \begin{bmatrix} 5 \\ 2 - 4i \end{bmatrix} e^{(0+4i)t}$

and by Euler's Theorem: $e^{(a\pm bi)t} = e^{at}(\cos bt \pm i \sin bt)$

we have $\mathbf{x}(t) = \begin{bmatrix} 5 \\ 2 - 4i \end{bmatrix} e^{0t}(\cos 4t + i \sin 4t)$

which can be written $\mathbf{x}(t) = \begin{bmatrix} 5 \cos 4t + 5i \sin 4t \\ 2 \cos 4t - 4i \cos 4t + 2i \sin 4t + 4 \sin 4t \end{bmatrix}$

	<u>Real</u>	<u>Imaginary</u>
Two solutions can be formed by separating the real and imaginary parts and discarding the i value.	$\mathbf{x}_1(t) = \begin{bmatrix} 5 \cos 4t \\ 2 \cos 4t + 4 \sin 4t \end{bmatrix}$	$\mathbf{x}_2(t) = \begin{bmatrix} 5 \sin 4t \\ -4 \cos 4t + 2 \sin 4t \end{bmatrix}$

Finally, using the formula for the general solution $x(t) = c_1 x_1(t) + c_2 x_2(t)$

we can form the general solutions

$$\begin{aligned} x_1(t) &= 5c_1 \cos 4t + 5c_2 \sin 4t \\ x_2(t) &= 2c_1 \cos 4t + 4c_1 \sin 4t - 4c_2 \cos 4t + 2c_2 \sin 4t \end{aligned}$$

using the initial values	$x_1(0) = 2$	$x_2(0) = 3$
we can solve for c_1 and c_2	$2 = 5c_1 \cos 0 + 5c_2 \sin 0$	$3 = 2c_1 \cos 0 + 4c_1 \sin 0 - 4c_2 \cos 0 + 2c_2 \sin 0$
	$2 = 5c_1$	$3 = 2c_1 - 4c_2$
	$c_1 = \frac{2}{5}$	$c_2 = -\frac{11}{20}$
we can now find particular solutions	$x_1(t) = 5\left(\frac{2}{5}\right)\cos 4t + 5\left(-\frac{11}{20}\right)\sin 4t$ and $x_2(t) = 2\left(\frac{2}{5}\right)\cos 4t + 4\left(\frac{2}{5}\right)\sin 4t - 4\left(-\frac{11}{20}\right)\cos 4t + 2\left(-\frac{11}{20}\right)\sin 4t$	
which simplifies to	$x_1(t) = 2 \cos 4t - \frac{11}{4} \sin 4t$ and $x_2(t) = 3 \cos 4t + \frac{1}{2} \sin 4t$	