The Fourier Series

Solving A Boundary Value Problem using the method of Separation of Variables and the Fourier Series

The Problem:

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$$\boxed{2u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0; \qquad u(0,t) = u(1,t) = 0, \quad u(x,0) = 4\sin\pi x\cos^3\pi x}$$

The Solution:

First of all we know that $\boxed{u_t = ku_{xx}}$ and since we are given $2u_t = u_{xx}$ we find that $\boxed{k = \frac{1}{2}}$.

L is the length and we are given $0 < x < 1$ so we know that $\boxed{L = 1}$.

We need to do some manipulation to the initial condition $u(x,0) = 4\sin p\ x\cos^3 p\ x$ so that it is in the form of sine only. We will use the trig identity $\boxed{\sin a\cos b = \frac{1}{2}[\sin(a+b)+\sin(a-b)]}$. There are other ways to convert the expression to sine only but this method is simple in that it uses only one identity.

We rewrite the initial condition as:	$u(x,0) = 4\sin\pi x\cos\pi x\cos^2\pi x$
We apply the trig identity to the term $\sin\pi x\cos\pi x$ with the result:	$u(x,0) = 2\sin 2\pi x\cos\pi x\cos\pi x$
We apply the identity a second time with the result:	$u(x,0) = \sin 3\pi x\cos\pi x + \sin\pi x\cos\pi x$
We again apply the identity, this time to both parts of the equation with the result:	$u(x,0) = \frac{1}{2}[\sin 4\pi x + \sin 2\pi x] + \frac{1}{2}[\sin 2\pi x + \sin 0]$
We now have the expression as a function of sine only which can be simplified to:	$u(x,0) = \frac{1}{2}\sin 4\pi x + \sin 2\pi x$

The Formula:

To gain our solution we have this formula: $\boxed{u_n(x,t) = X_n(x)\,T_n(t) = c_n e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}}$ where n is an integer or a series of integers and c is a constant or constants.

Filling in the known values for k and L we have $u_n(x,t) = c_n e^{-\frac{1}{2}n^2\pi^2 t} \sin n\pi x$.

We must select values for n and c to satisfy the initial condition $u(x,0) = \frac{1}{2}\sin 4\pi x + \sin 2\pi x$.

Notice that in the initial condition, $t = 0$ so the term $e^{-\frac{1}{2}n^2\pi^2 t}$ will become 1 and does not need our attention. The remaining term $c_n \sin n\pi x$ closely resembles the two terms in our initial condition such that we can pick out the values we need for c and n to satisfy the condition by inspection. In the first term, c = 1/2 and n = 4. In the second term c = 1 and n = 2.

The Answer:

So the solution is $\boxed{u_n(x,t) = \frac{1}{2}e^{-8\pi^2 t} \sin 4\pi x + e^{-2\pi^2 t} \sin 2\pi x}$