CONVOLUTION

As Applied to Linear Time-Invariant Systems

The **convolution integral** occurs frequently in the physical sciences. The convolution integral of two functions $f_1(t)$ and $f_2(t)$ is denoted symbolically by $f_1(t) * f_2(t)$.	$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

So what is happening graphically is that we are inverting the second function about the vertical axis, that is $f_2(-\tau)$. Then we shift the function right by t seconds so we have $f_2(t-\tau)$. Multiplying this by the first function yields a third function. Taking the integral yields the area under the graph of this third function. That's convolution.

CONTINUOUS-TIME SYSTEMS

The **Zero-state Response** can be written as the **convolution integral** of the Input and the Unit Impulse Response. If $f(t)$ and $h(t)$ are causal, the limits of integration are 0 to t.

Zero-state Response | Unit Impulse Response | Convolution Integral

$$y(t) = f(t) * h(t) = \int_0^t f(\tau)\, h(t-\tau) d\tau, \quad t \geq 0$$

Input

DISCRETE-TIME SYSTEMS

The **Zero-state Response** can be written as the **convolution sum** of the Input and the Unit Impulse Response:

Zero-state Response | Unit Impulse Response | Convolution Sum

$$y[k] = f[k] * h[k] = \sum_{m=-\infty}^{\infty} f[m]\, h[k-m]$$

Input

PROPERTIES OF CONVOLUTION

Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

Associative: $f_1(t) * \left[f_2(t) * f_3(t)\right] = \left[f_1(t) * f_2(t)\right] * f_3(t)$

Distributive: $f_1(t) * \left[f_2(t) * f_3(t)\right] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

Shift Property: If $f_1(t) * f_2(t) = c_1(t)$ then $f_1(t) * f_2(t-T) = c_1(t-T)$
$f_1(t-T) * f_2(t) = c_1(t-T)$

Convolution with an impulse: $f_1(t) * \delta(t) = f(t)$

Width Property: If $f_1(t)$ and $f_2(t)$ have durations of T_1 and T_2 respectively, then the duration of $f_1(t) * f_2(t)$ is $T_1 + T_2$.

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CONVOLUTION TABLE

	$f_1(t)$	$f_2(t)$	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
1	$f(t)$	$\delta(t-T)$	$f(t-T)$
2	$e^{\lambda t}\, u(t)$	$u(t)$	$\frac{-1}{\lambda}\left(1-e^{\lambda t}\right)u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}\, u(t)$	$e^{\lambda_2 t}\, u(t)$	$\frac{-1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)u(t), \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}\, u(t)$	$e^{\lambda t}\, u(t)$	$te^{\lambda t}\, u(t)$
6	$te^{\lambda t}\, u(t)$	$e^{\lambda t}\, u(t)$	$\frac{1}{2}t^2 e^{\lambda t}\, u(t)$
7	$t^n u(t)$	$e^{\lambda t}\, u(t)$	$\frac{n!}{\lambda^{n+1}} e^{\lambda t}\, u(t) - \sum_{j=0}^{n} \frac{n!}{\lambda^{j+1}(n-j)!} t^{n-j}\, u(t)$
8	$t^m u(t)$	$t^n u(t)$	$\frac{m!\,n!}{(n+m+1)!} t^{m+n+1}\, u(t)$
9	$te^{\lambda_1 t}\, u(t)$	$e^{\lambda_2 t}\, u(t)$	$\frac{1}{(\lambda_1-\lambda_2)^2}\left[e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}\right]u(t)$
10	$t^m e^{\lambda t}\, u(t)$	$t^n e^{\lambda t}\, u(t)$	$\frac{m!\,n!}{(n+m+1)!} t^{m+n+1} e^{\lambda t}\, u(t)$
11	$t^m e^{\lambda_1 t}\, u(t)$	$t^n e^{\lambda_2 t}\, u(t)$	$\sum_{j=0}^{m} \frac{(-1)^j m!(n+j)!}{j!(m-j)!(\lambda_1-\lambda_2)^{n+j+1}} t^{m-j} e^{\lambda_1 t}\, u(t)$ $+ \sum_{k=0}^{n} \frac{(-1)^k n!(m+k)!}{k!(n-k)!(\lambda_2-\lambda_1)^{m+k+1}} t^{n-k} e^{\lambda_2 t}\, u(t),$ $\lambda_1 \neq \lambda_2$
12	$e^{-\alpha t}\cos(\beta t+\theta)\, u(t)$	$e^{\lambda t}\, u(t)$	$\frac{\cos(\theta-\phi)e^{\lambda t} - e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}\, u(t)$ $\phi = \tan^{-1}\left[-\beta/(\alpha+\lambda)\right]$
13	$e^{\lambda_1 t}\, u(t)$	$e^{\lambda_2 t}\, u(-t)$	$\frac{1}{\lambda_2-\lambda_1}\left[e^{\lambda_1 t}\, u(t) + e^{\lambda_2 t} u(-t)\right],$ $\operatorname{Re}\lambda_2 > \operatorname{Re}\lambda_1$
14	$e^{\lambda_1 t}\, u(-t)$	$e^{\lambda_2 t}\, u(-t)$	$\frac{1}{\lambda_2-\lambda_1}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)u(-t)$