CONVOLUTION

As Applied to Linear Time-Invariant Systems

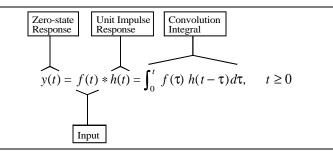
The **convolution integral** occurs frequently in the physical sciences. The convolution integral of two functions $f_1(t)$ and $f_2(t)$ is denoted symbolically by $f_1(t) * f_2(t)$.

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau$$

So what is happening graphically is that we are inverting the second function about the vertical axis, that is $f_2(-\tau)$. Then we shift the function right by t seconds so we have $f_2(t-\tau)$. Multiplying this by the first function yields a third function. Taking the integral yields the area under the graph of this third function. That's convolution.

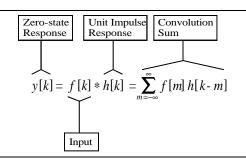
CONTINUOUS-TIME SYSTEMS

The **Zero-state Response** can be written as the **convolution integral** of the Input and the Unit Impulse Response. If f(t) and h(t) are causal, the limits of integration are 0 to t.



DISCRETE-TIME SYSTEMS

The **Zero-state Response** can be written as the **convolution sum** of the Input and the Unit Impulse Response:



PROPERTIES OF CONVOLUTION

Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$

Distributive: $f_1(t) * [f_2(t) * f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

Shift Property: If $f_1(t) * f_2(t) = c_1(t)$ then $f_1(t) * f_2(t-T) = c_1(t-T)$

 $f_1(t-T) * f_2(t) = c_1(t-T)$

Convolution with an impulse: $f_1(t) * \delta(t) = f(t)$

Width Property: If $f_1(t)$ and $f_2(t)$ have durations of T_1 and T_2 respectively, then the

duration of $f_1(t) * f_2(t)$ is $T_1 + T_2$.

CONVOLUTION TABLE

	$f_1(t)$	$f_2(t)$	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
1	f(t)	$\delta(t-T)$	f(t-T)
2	$e^{\lambda t} u(t)$	u(t)	$\frac{-1}{\lambda} \Big(1 - e^{\lambda t} \Big) u(t)$
3	u(t)	u(t)	tu(t)
4	$e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{-1}{\lambda_1 - \lambda_2} \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) u(t), \lambda_1 \neq \lambda_2$
5	$e^{\lambda t} u(t)$	$e^{\lambda t} u(t)$	$te^{\lambda t} u(t)$
6	$te^{\lambda t} u(t)$	$e^{\lambda t} u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^n u(t)$	$e^{\lambda t} u(t)$	$\frac{n!}{\lambda^{n+1}} e^{\lambda t} u(t) - \sum_{j=0}^{n} \frac{n!}{\lambda^{j+1} (n-j)!} t^{n-j} u(t)$
8	$t^m u(t)$	$t^n u(t)$	$\frac{m! n!}{(n+m+1)!} t^{m+n+1} u(t)$
9	$te^{\lambda_{l}t} u(t)$	$e^{\lambda_2 t} u(t)$	$\frac{1}{(\lambda_1 - \lambda_2)^2} \Big[e^{\lambda_2 t} - e^{\lambda_1 t} + (\lambda_1 - \lambda_2) t e^{\lambda_1 t} \Big] u(t)$
10	$t^m e^{\lambda t} u(t)$	$t^n e^{\lambda t} u(t)$	$\frac{m!n!}{(n+m+1)!}t^{m+n+1}e^{\lambda t}u(t)$
11	$t^m e^{\lambda_1 t} u(t)$	$t^n e^{\lambda_2 t} u(t)$	$\sum_{j=0}^{m} \frac{(-1)^{j} m! (n+j)!}{j! (m-j)! (\lambda_{1} - \lambda_{2})^{n+j+1}} t^{m-j} e^{\lambda_{1} t} u(t)$
			$+\sum_{k=0}^{n}\frac{(-1)^{k}n!(m+k)!}{k!(n-k)!(\lambda_{2}-\lambda_{1})^{m+k+1}}t^{n-k}e^{\lambda_{2}t}u(t),$
			$\lambda_1 \neq \lambda_2$
12	$e^{-\alpha t}\cos(\beta t + \theta)u(t)$	$e^{\lambda t} u(t)$	$\frac{\cos(\theta - \phi)e^{\lambda t} - e^{-\alpha t}\cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda)^2 + \beta^2}}u(t)$
			$\phi = \tan^{-1} \left[-\beta / (\alpha + \lambda) \right]$
13	$e^{\lambda_1 t} u(t)$	$e^{\lambda_2 t} u(-t)$	$\frac{1}{\lambda_2 - \lambda_1} \left[e^{\lambda_1 t} u(t) + e^{\lambda_2 t} u(-t) \right],$
14	$e^{\lambda_1 t} u(-t)$	$e^{\lambda_{2^t}}u(-t)$	Re $\lambda_2 > \text{Re } \lambda_1$ $\frac{1}{\lambda_2 - \lambda_1} \left(e^{\lambda_1 t} - e^{\lambda_2 t} \right) u(-t)$