THE LAPLACE TRANSFORM

Fundamentals of the Laplace Transform

THE LAPLACE TRANSFORM

The Laplace transform of a function *f*(*t*) is expressed symbolically as *F*(*s*), where *s* is a complex value.

$$\mathscr{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

The formula shown is called the **unilateral** or one-sided Laplace transform because the integration takes place over the interval from 0 to ∞; the **bilateral** or two-sided transform integrates from -∞ to ∞.

THE INVERSE LAPLACE TRANSFORM

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

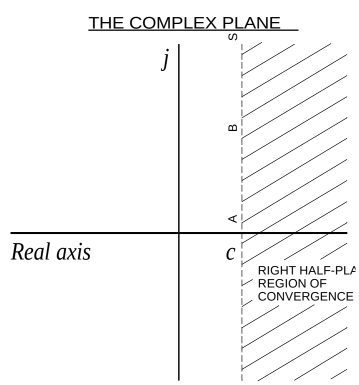
where *c* is the **abscissa of convergence** (defined later). The text says the use of this formula is too complicated for the scope of the book.

In my Differential Equations class, we had a substitute teacher one day that gave us this formula for the Inverse Laplace Transform. Normally you get the inverse Laplace transform from tables but this is a way to calculate. I don't know how it works but thought I would save it. He said that this and some other things that aren't found in current math textbooks are found in a 1935 book by Widder called "Advanced Calculus" which he recommends for engineers.

$$\mathscr{L}^{-1}[F(s)] = f(t) = \lim_{k\to\infty}\frac{(-1)^k}{k!}\times f^k\left(\frac{k}{t}\right)\times\left(\frac{k}{t}\right)^{k+1}$$

USING THE LAPLACE TRANSFORM

When finding the Laplace transform of a function, the result of performing the integration may contain a term such as $e^{-(s+a)t}$. It should be noted that as $t\to\infty$, this term does not necessarily go to infinity as well because of the complex variable *s*.



$$\lim_{t\to\infty} e^{-(s+a)t} = \begin{cases} 0, \text{when the real part of } s+a>0 \\ \infty, \text{when the real part of } s+a<0 \end{cases}$$

The solution concerns only the part of the complex plane where the real part of *s* + *a* in this example is greater than zero and this area is called the **region of convergence**. It is said to consist of the **right half-plane** of the complex plane bounded by the **abscissa of convergence**, *c*, which in this case is equal to the real part of *s* minus the variable *a*.

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TIME-DERIVATIVES OF THE LAPLACE TRANSFORM

The First Derivative: $F'(s) = sF(s) - \underbrace{f(0)}_{\text{initial condition}}$

The Second Derivative: $F''(s) = s^2F(s) - s\underbrace{f(0) - f'(0)}_{\text{initial conditions}}$

TIME-SHIFTING THE LAPLACE TRANSFORM

This formula represents a time-shift to the <u>right</u> (t_0 is positive).

$$f(t - t_0) \Leftrightarrow F(s)e^{-st_0} \qquad \mathscr{L}[f(t-t_0)] = \int_0^\infty f(t-t_0)e^{-st}e^{-st_0}dt\,, \qquad t_0 \geq 0$$

Delaying a signal by t_0 seconds is equivalent to multiplying its transform by e^{-st_0}. The time-shifting property is useful in finding the Laplace transform of piecewise continuous functions.

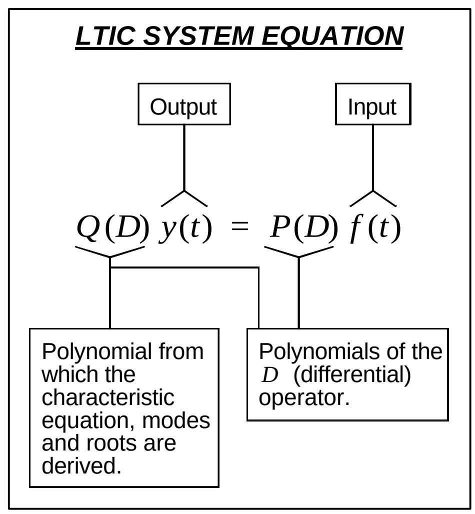
TIME-DOMAIN SOLUTIONS USING THE LAPLACE TRANSFORM

By taking the Laplace transform of an equation describing a linear time-invariant continuous-time (LTIC) system it is possible to simplify an equation of derivatives into an algebraic expression. The following substitutions are made:

$Y(s) \Leftrightarrow y(t)$, the zero-state response
$F(s) \Leftrightarrow f(t)$, the input function
$H(s) \Leftrightarrow P(t)/Q(t)$, or the ratio of $Y(s)/F(s)$ when all initial conditions are zero. The poles of $H(s)$ are the **characteristic roots** of the system. $H(s)$ is also the Laplace transform of the **unit impulse response** $h(t)$.

$$H(s) = \int_0^\infty h(t)e^{-st}dt$$

The transform of the equation is reduced to simplest form and then the inverse transform is taken using the table of Laplace transforms.



A TABLE OF LAPLACE TRANSFORMS

	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{As+B}{s^2+2as+c}$
	$r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$,	$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{c-a^2}}$, $\quad b=\sqrt{c-a^2}$
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$	$\frac{As+B}{s^2+2as+c}$, $\quad b=\sqrt{c-a^2}$

A TABLE OF LAPLACE TRANSFORM OPERATIONS

Operation	*f(t)*	*F(s)*
Addition	$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$
Scalar multiplication	$kf(t)$	$kF(s)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$
Time Integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
	$\int_{-\infty}^t f(t)\,dt$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^0 f(t)\,dt$
Time shift	$f(t-t_0)u(t-t_0)$	$F(s)e^{-st_0}, \quad t_0 \geq 0$
Frequency shift	$f(t)e^{s_0 t}$	$F(s-s_0)$
Frequency differentiation	$-tf(t)$	$\frac{dF(s)}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Scaling	$f(at), \ a \geq 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Frequency convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi j}F_1(s) * F_2(s)$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$ (poles of $sF(s)$ in LHP)