PROOF OF A SOLUTION

An example of the steps involved in proving that there is a solution to a statement and determining what that solution is.

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Definition: A solution is a function defined on $\Re_+$ that is differentiable for $t > 0$ which satisfies the problem statement.

Problem: Prove that there is a solution x for $\boxed{x'(t) + ax(t) = f(t), \ \ t \geq 0}$ and determine the solution.

Solution:

1. Assume that f is continuous.

2. Suppose that there exists a solution x. Then it can be written in this way because e^{-at} can never be zero and we haven't yet defined $q(t)$.

$$x(t) = q(t)e^{-at} \qquad \text{(eq. 2)}$$

3. Substitute $q(t)e^{-at}$ for $x(t)$ in the problem statement.

$$q'(t)e^{-at} + q(t)(-a)e^{-at} + aq(t)e^{-at} = f(t)$$

Then simplify:

$$q'(t)e^{-at} = f(t)$$

And rewrite:

$$q'(t) = e^{at} f(t)$$

4. Integrate from 0 to t.

$$\int_0^t q'(\tau)d\tau = \int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 4a)}$$

Then simplify:

$$q(t) - q(0) = \int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 4b)}$$

And rewrite, letting $q(0) = c$:

$$q(t) = \int_0^t e^{a\tau} f(\tau)d\tau + c \qquad \text{(eq. 4c)}$$

5. So if there is a solution x, we must have:

$$x(t) = q(t)e^{-at} \qquad \text{(eq. 2)}$$

with:

$$q(t) = \int_0^t e^{a\tau} f(\tau)d\tau + c \qquad \text{(eq. 4c)}$$

6. Substituting $q(t)$ as given in (eq. 4c) into the equation (eq. 2) we have:

$$x(t) = ce^{-at} + e^{-at}\int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 6a)}$$

$$x(t) = ce^{-at} + \int_0^t e^{-at}e^{a\tau} f(\tau)d\tau$$

If there is a solution x, then the solution is:

$$x(t) = ce^{-at} + \int_0^t e^{-a(t-\tau)} f(\tau)d\tau \qquad \text{(eq. 6b)}$$

7. Setting $t = 0$ we see that $c = x(0)$.

$$x(0) = ce^0 + 0$$

8. So, taking (eq. 6a):

$$x(t) = ce^{-at} + e^{-at}\int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 6a)}$$

And differentiating:

$$x'(t) = -ace^{-at} + e^{-at}e^{at} f(t) - ae^{-at}\int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 8a)}$$

This simplifies to:

$$x'(t) = -ace^{-at} - ae^{-at}\int_0^t e^{a\tau} f(\tau)d\tau \qquad \text{(eq. 8b)}$$

9. So from the problem statement and (eq. 6b) and (eq. 8b) we have:

$$x'(t) + ax(t) =$$
$$-ace^{-at} - ae^{-at}\int_0^t e^{a\tau} f(\tau)d\tau + f(t) + ace^{-at} + a\int_0^t e^{-a(t-\tau)} f(\tau)d\tau$$
$$= f(t)$$

10. This shows that given:

$$x'(t) + ax(t) = f(t), \quad x(0) = c, \quad t \geq 0$$

there is exactly one solution called x and x is given by:

$$\boxed{x(t) = ce^{-at} + \int_0^t e^{-a(t-\tau)} f(\tau)d\tau, \quad t \geq 0} \qquad \text{(eq. 6b)}$$