THE Z-TRANSFORM

Fundamentals of the Z-Transform

The z-transform simplifies the analysis of linear time-invariant discrete-time (LTID) systems. It converts equations with integrals and derivatives into algebraic equations. The z-transform method of analyzing discrete-time systems is comparable to the Laplace transform method of analyzing continuous-time systems.

THE UNILATERAL Z-TRANSFORM

The unilateral z-transform is capable of analyzing only causal systems with causal inputs (signals starting at $k = 0$).

$$F[z] \equiv \sum_{k=0}^{\infty} f[k] z^{-k}$$

where: $F[z]$ is the z-transform
z is complex in general
$f[k]$ is a discrete-time signal

THE INVERSE Z-TRANSFORM

$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1}\, dz$$

where: $\oint$ indicates integration around a circular path in the complex plane centered at the origin
$F[z]$ is the z-transform
z is complex in general
$f[k]$ is a discrete-time signal

THE BILATERAL Z-TRANSFORM

The bilateral z-transform is capable of analyzing both causal and non-causal systems.

$$F[z] \equiv \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$

It is helpful when working these problems to know that: $1 + x + x^2 + x^3 + \cdots + x^\infty = \dfrac{1}{1-x}$

A TABLE OF Z-TRANSFORMS

	$f[k]$	$F[z]$
1	$\delta[k-j]$	z^{-j}
2	$u[k]$	$\frac{z}{z-1}$
3	$ku[k]$	$\frac{z}{(z-1)^2}$
4	$k^2u[k]$	$\frac{z(z+1)}{(z-1)^3}$
5	$k^3u[k]$	$\frac{z(z^2+4x+1)}{(z-1)^4}$
6	$\gamma^{k-1}u[k-1]$	$\frac{1}{z-\gamma}$
7	$\gamma^k u[k]$	$\frac{z}{z-\gamma}$
8	$k\gamma^k u[k]$	$\frac{\gamma z}{(z-\gamma)^2}$
9	$k^2\gamma^k u[k]$	$\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\frac{k(k-1)(k-2)\cdots(k-m+1)}{\gamma^m m!}\gamma^k u[k]$	$\frac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^k \cos\beta k\, u[k]$	$\frac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^k \sin\beta k\, u[k]$	$\frac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^k \cos(\beta k+\theta)u[k]$	$\frac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^k \cos(\beta k+\theta)u[k]$	$\frac{(0.5re^{j\theta})z}{z-\gamma}+\frac{(0.5re^{-j\theta})z}{z-\gamma^*},\qquad \gamma=\lvert\gamma\rvert e^{j\beta}$
12c	$r\lvert\gamma\rvert^k \cos(\beta k+\theta)u[k]$	$\frac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}} \qquad \beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}, \qquad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert-a^2}}$$

A TABLE OF Z-TRANSFORM OPERATIONS

Operation	$f[k]$	$F[z]$
Addition	$f_1[k]+f_2[k]$	$F_1[z]+F_2[z]$
Scalar multiplication	$af[k]$	$aF[z]$
Right-shift	$f[k-m]u[k-m]$	$\frac{1}{z^m}F[z]$
	$f[k-m]u[k]$	$\frac{1}{z^m}F[z]+\frac{1}{z^m}\sum_{k=1}^{m} f[-k]z^k$
	$f[k-1]u[k]$	$\frac{1}{z}F[z]+f[-1]$
	$f[k-2]u[k]$	$\frac{1}{z^2}F[z]+\frac{1}{z}f[-1]+f[-2]$
Left-shift	$f[k+m]u[k]$	$z^m F[z]-z^m\sum_{k=0}^{m-1} f[k]z^{-k}$
	$f[k+1]u[k]$	$zF[z]-zf[0]$
	$f[k+2]u[k]$	$z^2F[z]-z^2f[0]-zf[1]$
	$f[k+3]u[k]$	$z^3F[z]-z^3f[0]-z^2f[1]-zf[2]$
Multiplication by γ^k	$\gamma^k f[k]u[k]$	$F\left[\frac{z}{\gamma}\right]$
Multiplication by k	$kf[k]u[k]$	$-z\frac{d}{dz}F[z]$
Convolution	$f_1[k]*f_2[k]$	$F_1[z]F_2[z]$
Initial value	$f[0]$	$\lim_{s\to\infty} F[z]$
Final value	$\lim_{N\to\infty} F[N]$	$\lim_{z\to 1}(z-1)F[z]$,
		poles of $(z-1)F[z]$ inside the unit circle