THE GRAM-SCHMIDT PROCESS and QR FACTORIZATION

Given a matrix M with linearly independent columns, the following steps will yield an orthonormal $n \times m$ matrix Q and an upper triangular $m \times m$ matrix R such that $M = QR$. Consider the form below where M is the 2-column matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2]$.

$$M = [\mathbf{v}_1 \quad \mathbf{v}_2] = [\mathbf{w}_1 \quad \mathbf{w}_2]\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = QR$$

We must find r_{11}, $\mathbf{w}_1$ r_{12}, r_{22}, and $\mathbf{w}_2$ preferably in that order.

$r_{11} = \lVert \mathbf{v}_1 \rVert$	$\lVert \mathbf{v}_1 \rVert$ means "the norm of $\mathbf{v}_1$", which is the length of the vector. For example if $\mathbf{v}_1$ were composed of the elements a, b, and c, then $\lVert \mathbf{v}_1 \rVert = \sqrt{a^2 + b^2 + c^2}$.

$$\mathbf{w}_1 = \frac{1}{r_{11}} \mathbf{v}_1$$

$r_{12} = \mathbf{w}_1 \cdot \mathbf{v}_2$	For example if $\mathbf{w}_1$ were composed of the elements d, e, f and $\mathbf{v}_2$ were composed of the elements a, b, and c, then:	$\mathbf{w}_1 \cdot \mathbf{v}_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = da + eb + fc$

$$r_{22} = \lVert \mathbf{v}_2 - r_{12}\mathbf{w}_1 \rVert$$

$$\mathbf{w}_2 = \frac{1}{r_{22}} (\mathbf{v}_2 - r_{12}\mathbf{w}_1)$$

If M is a 3-column matrix, we can use the above values and continue by finding r_{13}, r_{23}, r_{33}, and $\mathbf{w}_3$.

$$M = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3]\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = QR$$

$$r_{13} = \mathbf{w}_1 \cdot \mathbf{v}_3 \qquad r_{23} = \mathbf{w}_2 \cdot \mathbf{v}_3 \qquad r_{33} = \lVert \mathbf{v}_3 - r_{13}\mathbf{w}_1 - r_{23}\mathbf{w}_2 \rVert \qquad \mathbf{w}_3 = \frac{1}{r_{33}} (\mathbf{v}_3 - r_{13}\mathbf{w}_1 - r_{23}\mathbf{w}_2)$$

APPLICATIONS OF THE GRAM-SCHMIDT PROCESS

This process can be used to find an **orthonormal vector** as well. For example if we are given 2 orthonormal vectors in $\Re^3$, and wish to find the third, just pick any third vector that is linearly independent of the other two and form a 3×3 matrix. Do a QR factorization on this matrix. In the result *Q*, will be the two original vectors and the third (unique) orthonormal vector.

When working with **inner product spaces**, substitute inner product notation where dot products are found in the Gram-Schmidt process, i.e. where $\mathbf{w}_1 \cdot \mathbf{v}_2$ is found, substitute $\langle \mathbf{w}_1, \mathbf{v}_2 \rangle$. (The dot product is a type of inner product.)